Chapter 01 — Basic Maths and Logarithm

CONTENT

BASIC MATHS

1. NUMBER SYSTEM :

Natural Numbers : N = {1, 2, 3....}
Whole Numbers : W = {0, 1, 2, 3.....}
Integers : I = Z {,........–3, –2, –1, 0, 1, 2, 3.....}
Positive Integers : $I^+ = Z^+$ {1, 2, 3 ... }
Negative Integers : $I^- = Z^-$ {, ... –3, –2, –1}
Non-negative Integers : {0, 1, 2, 3........}
Non-positive Integers : {,.... ,–3, –2, –1, 0}
Even Integers = {....–6, –4, –2, 0, 2, 4, 6 ...}
Odd Integers = {–5, –3, –1, 1, 3, 5 ......}
None-zero integers Z_0 = {,........–3, –2, –1, 1, 2, 3.....}

DETECTIVE MIND

(i) Zero is neither positive nor negative.
(ii) Zero is even number.
(ii) Positive means > 0.
(iv) Non-negative means ≥ 0

Surds

If a is not a perfect n^{th} power, then $\sqrt[n]{a}$ is called a surd of the n^{th} order.

In an expression of the form $\frac{a}{\sqrt{b}+\sqrt{c}}$, the denominator can be rationalized by multiplying numerator and the denominator by $\sqrt{b}-\sqrt{c}$ which is called the conjugate of $\sqrt{b}+\sqrt{c}$. If $x+\sqrt{y}=a+\sqrt{b}$ where x, y, a, b are rationals, then x = a and y = b.

2. FRACTION $\left(\frac{p}{q}\right)$:

(1) Proper Fraction $=\frac{3}{5}$: $N^r < D^r$

(2) Improper Fraction $=\frac{5}{3} : N^r > D^r$

(3) Mixed Fraction : $2\frac{3}{5}$

(4) Complex Fraction : $\frac{\frac{2}{3}}{\frac{5}{6}}$

(5) Continued Fraction : $2+\cfrac{2}{2+\cfrac{2}{+.....}}$

3. DIVISIBILITY RULES :

Divisible by 2	Last digit 0, 2, 4, 6, 8
Divisible by 3	Sum of digits divisible by 3(Remainder will be same when number is divided by 3 or sum of digits is divided by 3)
Divisible by 4	Last two digits divisible by 4 (Remainder will be same whether we divide the number or its last two digits)
Divisible by 5	Last digit 0 or 5
Divisible by 6	Divisible by 2 and 3 simultaneously.
Divisible by 8	Last three digits is divisible by 8 (Remainder will be same whether we divide the number or its last three digits)
Divisible by 9	Sum of digits divisible by 9. (Remainder will be same when number is divided by 9 or sum of digit is divided by 9)
Divisible by 10	Last digits 0
Divisible by 11	(Sum of digits at even places) – (sum of digits at odd places) = 0 or divisible by 11

4. LCM AND HCF :

(1) HCF is the highest common factor between any two or more numbers or algebraic expressions. When dealing only with numbers, it is also called "Greatest common divisor" (GCD).

(2) LCM is the lowest common multiple of two or more numbers or algebraic expressions.

(3) The product of HCF and LCM of two numbers (or expressions) is equal to the product of the numbers.

(4) LCM of $\left(\frac{a}{b}, \frac{p}{q}, \frac{l}{m}\right) = \frac{\text{L.C.M. of } (a, p, l)}{\text{H.C.F. of } (b, q, m)}$

5. FACTORIZATION :

Formulae :

(1) $(a \pm b)^2 = a^2 \pm 2ab + b^2 = (a \mp b)^2 \pm 4ab$

(2) $a^2 - b^2 = (a+b)(a-b)$;

If $a^2 - b^2 = 1$ then $a + b = \frac{1}{a-b}$

(3) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(4) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(5) $a^3 + b^3 = (a+b)(a^2 - ab + b^2) = (a+b)^3 - 3ab(a+b)$

(6) $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)^3 + 3ab(a-b)$

(7) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

(8) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(9) $(a + b + c)^3 = a^3+b^3+c^3 + 3(a + b)(b + c)(c + a)$

(10) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

6. REMAINDER THEOREM :

If a polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divided by $x-p$, then the remainder is obtained by putting $x = p$ in the polynomial.

7. FACTOR THEOREM :

A polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divisible by $x-p$, if the remainder is zero

i.e. if $a_1p^n + a_2p^{n-1} + \ldots + a_n = 0$ then $x - p$ will be a factor of polynomial.

8. RATIO AND PROPORTION :

(1) If $\frac{a}{b} = \frac{c}{d}$, then : $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo); $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo);

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo); $\frac{a}{c} = \frac{b}{d}$ (alternendo);

$\frac{b}{a} = \frac{d}{c}$ (invertendo)

(2) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each ratio $= \left(\frac{a^n + c^n + e^n}{b^n + d^n + f^n}\right)^{\frac{1}{n}}$

SOLVED EXAMPLES

Example: 1 Prove that $\log_3 5$ is irrational.

Solution: let $\log_3 5$ is rational

$\therefore \log_3 5 = \frac{p}{q}$; where p and q are co- prime numbers.

$\Rightarrow 3^{p/q} = 5 \Rightarrow 3^p = 5^q$ which is not possible, hence our assumption is wrong and $\log_3 5$ is irartional.

Example: 2 Simplify: $\frac{12}{3+\sqrt{5}-2\sqrt{2}}$

Solution: the expression = $\frac{12\left(3+\sqrt{5}+2\sqrt{2}\right)}{\left(3+\sqrt{5}\right)^2-\left(2\sqrt{2}\right)^2}=\frac{12\left(3+\sqrt{5}+2\sqrt{2}\right)}{6+6\sqrt{5}}$

$=\frac{2\left(3+\sqrt{5}+2\sqrt{2}\right)\left(\sqrt{5}-1\right)}{\left(\sqrt{5}+1\right)\times\left(\sqrt{5}-1\right)}=\frac{2\left(2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2}\right)}{4}\Rightarrow 1+\sqrt{5}+\sqrt{10}-\sqrt{2}$

Example: 3 **Find the square root of** $7+2\sqrt{10}$

Solution: Let $\sqrt{7+2\sqrt{10}}=\sqrt{x}+\sqrt{y}$ $\Rightarrow$ Squaring, $x+y+2\sqrt{xy}=7+2\sqrt{10}$

hence, x + y = 7 and xy = 10. these two relation give

x =5, y = 2. Hence $\sqrt{7+2\sqrt{10}}=\sqrt{5}+\sqrt{2}$

Example: 4 Find p and q so that (x+2) and (x–1) may be factors of the polynomial $f(x)=x^3+10x^2+px+q$.

Solution: Since $(x+2)$ is factor $\Rightarrow f(-2)$ must be zero

$\therefore -8+40-2p+q=0$ ...(1)

Since (x – 1) is factor $\Rightarrow f(1)$ must be zero

$\therefore 1+10+p+q=0$ ... (2)

form (1) and (2), by solving we get p = 7 and q = –18

Example: 5 A polynomial in x of second degree which will vanish at x = 4 and x = 2 and will have value 3 when x = 5 respectively, is in the form of ax^2+bx+c. Find value of a + b + c

Solution : Hence p(x) vanish at x = 2 and x = 4 , so

p(x) = a(x – 2) (x – 4)

Given that p(5) = 3 , p(5) = a (5 – 2) (5 – 4) = 3, 3a – 3 = 0

a = 1 so,

$p(x)=(x-2)(x-4)=x^2-6x+8.\Rightarrow a+b+c=3$

Example: 6 Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4\left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

Solution : $a^{9\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)4} \cdot a^{9\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)4} = a^2 \cdot a^2 = a^4$

Example: 7 Find rational numbers a and b, such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a + b\sqrt{5}$

Solution : $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} = a + b\sqrt{5}$

$\frac{61+24\sqrt{5}}{-29} = a + b\sqrt{5} \Rightarrow a = -\frac{61}{29}, b = -\frac{24}{29}$

Example: 8 If x : y = 3 : 4, then find the ratio of 7x – 4y : 3x + y

Solution : $\frac{x}{y} = \frac{3}{4}$

$\therefore x = 3y \Rightarrow x = \frac{3}{4}y$

Now $\frac{7x-4y}{3x+y} = \frac{7 \cdot \frac{3}{4}y - 4y}{3 \cdot \frac{3}{4}y + y}$ (putting the value of x)

$= \frac{\frac{21}{4}y - 4y}{\frac{9}{4}y + y} = \frac{5y}{13y} = \frac{5}{13}$

Example: 9 If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, then show that $\frac{x^3+a^3}{x^2+a^2} + \frac{y^3+b^3}{y^2+b^2} + \frac{z^3+c^3}{z^2+c^2} = \frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$.

Solution : $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ k (constant)

$\therefore$ x = ak; y = bk; z = ck

Substituting these values of x, y, z in the given expression

$$\frac{x^3+a^3}{x^2+a^2} + \frac{y^3+b^3}{y^2+b^2} + \frac{z^3+c^3}{z^2+c^2} = \frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$$

we obtain

$$\text{L.H.S.} = \frac{a^3k^3+a^3}{a^2k^2+a^2} + \frac{b^3k^3+b^3}{b^2k^2+b^2} + \frac{c^3k^3+c^3}{c^2k^2+c^2} = \frac{a^3(k^3+1)}{a^2(k^2+1)} + \frac{b^3(k^3+1)}{b^2(k^3+1)} + \frac{c^3(k^3+1)}{c^2(k^2+1)}$$

$$= \frac{a(k^3+1)}{k^2+1} + \frac{b(k^3+1)}{k^2+1} + \frac{c(k^3+1)}{k^2+1} = \frac{(k^3+1)}{(k^2+1)} \cdot (a+b+c)$$

$$\text{Now R.H.S} = \frac{(ak+bk+ck)^3+(a+b+c)^3}{(ak+bk+ck)^2+(a+b+c)^2} = \frac{k^3(a+b+c)^3+(a+b+c)^3}{k^2(a+b+c)^2+(a+b+c)^2}$$

$$= \frac{(k^3+1)(a+b+c)^3}{(k^2+1)(a+b+c)^2} = \frac{(k^3+1)}{(k^2+1)} \cdot (a+b+c)$$

L.H.S. = R.H.S.

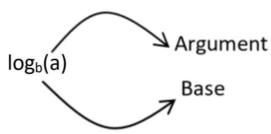
LOGARITHM

9. DEFINITION OF LOGARITHM

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent.

We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$.

Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number.



Limitations of logarithm: $\log_a N$ is defined only when

(i) $N > 0$ (ii) $a > 0$ (iii) $a \neq 1$

DETECTIVE MIND

(i) For a given value of N, $\log_a N$ will give us a unique value.

(ii) Logarithm of zero does not exist.

(iii) Logarithm of negative reals are not defined in the system of real numbers.

10. FUNDAMENTAL IDENTITIES :

Using the basic definition of logarithm we have 3 important deductions :

(1) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(2) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(3) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

(4) $\mathbf{N = (a)^{\log_a N}}$ e.g. $2^{\log_2 7} = 7$

11. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where $a > 0$, $a \neq 1$ and x is any real number, then-

(1) $\log_a mn = \log_a m + \log_a n$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(3) $\log_a m^x = x \log_a m$

12. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \dfrac{\log_a m}{\log_a b}$ where a > 0, a ≠ 1, b > 0, b ≠ 1

DETECTIVE MIND

(i) $\log_b a \cdot \log_a b = \dfrac{\log a}{\log b} \cdot \dfrac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \dfrac{1}{\log_a b}}$.

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) **Base power formula :** $\log_{a^k} m = \dfrac{1}{k}\log_a m$

13. TYPES OF LOGARITHM

Types of Logarithm

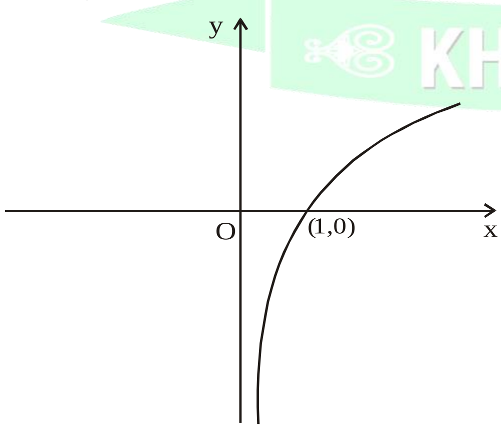
Natural Log
Log having base 'e'.
e → Napier's constant
→ irrational Number
→ 2.7 – 2.8
$\log_e(a) = \ell n\ a$

Common Log
Log having base '10'.
$\log_{10}(a) = \log a$

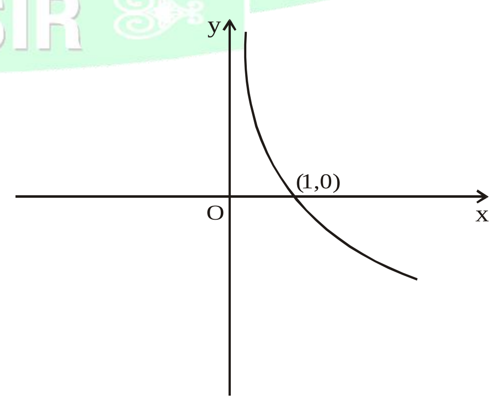
➢ Relation between log x and ln x : ln x = 2.303 log x (approximate)

14. GRAPHS:

y = $\log_a x$:



When a > 1



When 0 < a < 1

Points to remember :

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$; $\log_2\left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

15. LOGARITHMIC EQUATIONS

- $\text{Log}_b{}^a = p \Rightarrow a = b^p$
- $\text{Log}_b a = \log_b c \Rightarrow a = c$
- $\text{Log}_b a = \log_c a \Rightarrow b = c$

16. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

DETECTIVE MIND

(i) The mantissa part of logarithm of a number is always positive $(0 \leq m < 1)$

(ii) If the characteristic of $\log_{10} N$ be n, then the number of digits in N is (n + 1)

(iii) If the characteristic of $\log_{10} N$ be (–n), then there exist (n – 1) zeros after decimal in N.

17. ANTILOGARITHM :

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, **log n = m $\Leftrightarrow$ n = antilog m**

SOLVED EXAMPLES

Example: 10 The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is -

(1) 4 (2) 3 (3) 2 (4) 1

Solution : $1 + \log_b\{1 + \log_c(1 + \log_p N)\} = a^0 = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0 \Rightarrow 1 + \log_c(1 + \log_p N) = 1$

$\Rightarrow \log_c(1 + \log_p N) = 0 \Rightarrow 1 + \log_p N = 1$

$\Rightarrow \log_p N = 0 \Rightarrow N = 1$

Example: 11 If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$\Rightarrow \therefore \text{LHS} = \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1} = \text{RHS}$

Example: 12 Prove that $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$

Solution : $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

$= \log\left(\frac{16}{15}\right)^7 + \log\left(\frac{25}{24}\right)^5 + \log\left(\frac{81}{80}\right)^3 = \log\left(\left(\frac{16}{15}\right)^7 \times \left(\frac{25}{24}\right)^5 \times \left(\frac{81}{80}\right)^3\right)$

$= \log\left[\left(\frac{2^4}{3\times5}\right)^7 \times \left(\frac{5^2}{2^3\times3}\right)^5 \times \left(\frac{3^4}{2^4\times5}\right)^3\right] = \log\left[\frac{2^{28}}{3^7\times5^7} \times \frac{5^{10}}{2^{15}\times3^5} \times \frac{3^{12}}{2^{12}\times5^3}\right]$

$= \log\left[2^{28-15-12} \times 5^{10-7-3} \times 3^{12-7-5}\right] = \log(2^1 \times 5^0 \times 3^0) = \log 2$

Example: 13 If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

L.H.S. = $\log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b)$ = R.H.S.

Example: 14 If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$) -

(1) $\frac{1}{p} + \frac{1}{q}$ (2) $\frac{1}{2p} + \frac{1}{q}$ (3) $\frac{1}{p} + \frac{1}{2q}$ (4) $\frac{1}{2p} + \frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p} + \frac{1}{q}$

Example: 15 If a, b, c are distinct positive real numbers different from 1 such that

$(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c \,.\, \log_b c - \log_c c) = 0$, then abc is equal to -

(1) 0 (2) e (3) 1 (4) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \,.\, \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c}+\frac{\log b}{\log a}.\frac{\log b}{\log c}+\frac{\log c}{\log a}.\frac{\log c}{\log b}=3$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Example: 16 Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$

= 625 + 216 + 49 = 890.

Example: 17 $\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c}$ has the value equal to

(1) abc (2) $\frac{1}{abc}$ (3) 0 (4) 1

Solution : $\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c} = \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$

$= \log_{abc} b + \log_{abc} c + \log_{abc} a = \log_{abc} abc = 1$

Example: 18 Let $x = 2^{\log 3}$ and $y = 3^{\log 2}$ where base of the logarithm is 10, then which one of the following holds good?

(1) 2x < y (2) 2y < x (3) 3x = 2y (4) y = x

Solution : $x = 2^{\log 3}$, $y = 3^{\log 2} = 2^{\log 3} = x$

Example: 19 The ratio $\frac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$ simplifies to :

(1) $a^2 - a - 1$ (2) $a^2 + a - 1$ (3) $a^2 - a + 1$ (4) $a^2 + a + 1$

Solution : $y = \frac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$

$\Rightarrow$ but $2^{\log_{2^{1/4}} a} = 2^{4\log_2 a} = a^4$, $7^{4\log_{49} a} = 7^{2\log_7 a} = a^2$, $3^{\log_{27}(a^2+1)^3} = 3^{\log_3 (a^2+1)} = a^2 + 1$

$\therefore y = \frac{a^4 - (a^2 + 1 + 2a)}{a^2 - a - 1} = \frac{a^4 - (a+1)^2}{a^2 - a - 1} = a^2 + a + 1$

Example: 20 $10^{\log_p(\log_q(\log_r x))} = 1$ and $\log_q\left(\log_r(\log_p x)\right) = 0$ then 'p' equals

(1) $r^{q/r}$ (2) rq (3) 1 (4) $r^{r/q}$

Solution : $\log_p(\log_q(\log_r x)) = 0 \Rightarrow \log_q(\log_r x) = 1$

$\Rightarrow \log_r x = q \qquad \Rightarrow x = r^q$ ....(i)

and $\log_q(\log_r(\log_p x)) = 0$

$\Rightarrow \log_r(\log_p x) = 1 \qquad \Rightarrow \log_p x = r$

$\Rightarrow x = p^r$ ....(ii)

from (i) and (ii) $p^r = r^q$

(i) (ii) $p^r = r^q$

$\Rightarrow p = r^{q/r}$

Example:21 The sum of all the solutions to the equation $2\log_{10} x - \log_{10}(2x - 75) = 2$ is

(1) 30 (2) 350 (3) 75 (4) 200

Solution : $2\log_{10}x - \log_{10}(2x - 75) = 2 \Rightarrow \dfrac{x^2}{2x-75} = 10^2 = 100$

$\Rightarrow x^2 - 200x + 7500 = 0 \quad \Rightarrow \quad x = 50, x = 150$

sum = 200

Example:22 If $3^{2\log_3 x} - 2x - 3 = 0$, then the number of values of 'x' satisfying the equation is

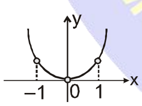
(1) zero (2) 1 (3) 2 (4) more than 2

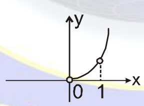
Solution : $x^2 - 2x - 3 = 0$

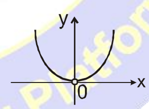
$(x - 3)(x + 1) = 0 \quad \Rightarrow x = 3, x = -1 \quad$ but $x \neq -1$

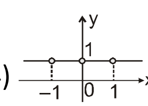
$\therefore\ x = 3.$

Example:23 The correct graph of $y = x^{\log_x x^2}$ is

(1) 

(2) 

(3) 

(4) 

Solution : Clearly Domain is $x > 0$ and $x \neq 1$

$x > 0\ x \neq 1$

Example: 24 Solve $\log_2(x^2) + \log_2(x + 2) = 4$

Solution : $\log_2(x^2(x + 2)) = 4 \Rightarrow x^3 + 2x^2 - 16 = 0 \qquad \Rightarrow (x-2)\underbrace{(x^2+4x+8)}_{D<0} = 0 \ \ x = 2$

19. INTERVALS :

Intervals are basically subsets of R. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

(1) Open interval : (a, b) = {x : a < x < b} i.e. end points are not included.

(2) Closed interval : [a, b] = {x : a $\leq$ x $\leq$ b} i.e. end points are also included.

This is possible only when both a and b are finite.

(3) Semi open or semi closed interval : $(a, b] = \{x : a < x \le b\}$; $[a, b) = \{x : a \le x < b\}$

(4) The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$ (ii) $[a, \infty) = \{x : x \ge a\}$

(iii) $(-\infty, b) = \{x : x < b\}$ (iv) $(-\infty, b] = \{x : x \le b\}$

(v) $(-\infty, \infty) = R$

DETECTIVE MIND

(i) For some particular values of x, we use symbol { } e.g. If x = 1, 2 we can write it as $x \in \{1, 2\}$

(ii) If their is no values of x, then we say $x \in \phi$ (null set)

20. INEQUALITY :

In Mathematics, a relationship between two expressions or values that are not equal to each other is called 'Inequality'.

Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' form an inequality.

'<', '>' are called strict inequality and '≤' , '≥' are called as slack inequality

(1) Numerical inequalities ⇒ examples $2 < 8$; $6 > 3$

(2) Literal inequalities ⇒ examples $x < 5$; $y > 2$; $x \ge 3$, $y \le 4$

(3) Double inequalities ⇒ examples $3 < 5 < 7$ and $3 < x < 5$

(1) LINEAR INEQUALITY

To solve these inequalities, keep the following rules in mind:

Rule 1: Equal numbers may be added to or subtracted from both sides of an inequality without affecting the sign of inequality.

Rule 2: both sides of an inequality can be multiplied or divided by the same positive number without affecting the sign of inequality, but when both sides are multiplied or divided by a negative number, then the sign of inequality is reversed (i.e., '<' becomes '> ,'≤' becomes ≥').

(2) HIGHER ORDER INEQUALITY

Method of intervals OR Wavy curve method

Let $f(x) = \left(\dfrac{(x-a_1)^{k_1}(x-a_2)^{k_2} --- (x-a_n)^{k_n}}{(x-b_1)^{r_1}(x-b_2)^{r_2} --- (x-b_n)^{r_n}}\right)$ ... (i)

Where $k_1, k_2 \ldots\ldots k_n$ and $r_1, r_2 \ldots\ldots\ldots r_n \in N$ and $a_1, a_2 \ldots\ldots a_n$ and $b_1, b_2 \ldots\ldots b_n$ are real numbers.

To solve these inequalities, keep the following rules in mind:

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

Steps : -

(1) First we find the zeros and poles of the function.

(2) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.

(3) Determine sign of the function in any of the interval and then alternates the sign in the neighbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

(4) Thus we consider all the intervals. The solution of the $g(x) > 0$ is the union of the intervals in which we have put the plus sign and the solution of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

(3) MODULUS INEQUALITY

To solve these inequalities, keep the following rules in mind:

- If $|x| < a$, then $-a < x < a$
- If $|x| > a$, then either $x > a$ or $x < -a$
- If $|x - r| < a$, then $r - a < x < r + a$
- If $|x - r| > a$, then either $x > r + a$ or $x < r - a$.

(4) SQUARE ROOT INEQUALITY

To solve these inequalities, keep the following rules in mind:

- The inequality $\sqrt{f(x)} < g(x)$, is equivalent to the following system

  $f(x) < g^2(x)$ & $f(x) \geq 0$ & $g(x) \geq 0$

- The inequality $\sqrt{f(x)} > g(x)$, is equivalent to the following system

  $g(x) \leq 0$ & $f(x) \geq 0$ or $g(x) \geq 0$ & $f(x) > g^2(x)$

(E) LOGARITHMIC INEQUALITY

To solve these inequalities, keep the following rules in mind:

- $\log_a x > p \Rightarrow \begin{cases} x > a^P & \text{if } a > 1 \\ x < a^P & \text{if } 0 < a < 1 \end{cases}$

- $\log_a x > \log_a y \Rightarrow \begin{cases} x > y & \text{if } a > 1 \\ x < y & \text{if } 0 < a < 1 \end{cases}$

SOLVED EXAMPLES

Example: 25 Solve the inequality $\frac{x}{2} \geq \frac{5x-2}{3} - \frac{7x-3}{5}$.

Solution : Given, $\frac{x}{2} \geq \frac{5x-2}{3} - \frac{7x-3}{5}$

Take LCM at right side

$\frac{x}{2} \geq \frac{5(5x-2)-3(7x-3)}{15}$

$\frac{x}{2} \geq \frac{25x-10-21x+9}{15}$

$\frac{x}{2} \geq \frac{4x-1}{15}$

$15x \geq 2(4x-1)$

$7x \geq -2$

$x \geq \frac{-2}{7}$

So the solution is $\left[\frac{-2}{7}, \infty\right)$.

Example: 26 Solve $|5-x| \leq 2$

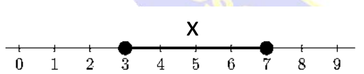
Solution : $-2 \leq 5-x \leq 2$

$-7 \leq -x \leq -3$

$7 \geq x \geq 3$

It is better to write this as: $3 \leq x \leq 7$.

Here's the graph of the solution:



Example: 27 Solve $\left|\frac{4x}{3} - 5\right| \geq 7$

Solution : $\left|\frac{4x}{3} - 5\right| \geq 7$

$\frac{4x}{3} - 5 \leq -7$ or $\frac{4x}{3} - 5 \geq 7$

$4x - 15 \leq -21$ $\quad$ $4x - 15 \geq 21$

$4x \leq -6$ $\quad$ $4x \geq 36$

$x \leq -1.5$ $\quad$ $x \geq 9$

So the solution is $x \in (-\infty, -1.5] \cup [9, \infty)$

Example: 28 $\sqrt{2x-5} + 7 < 12$

Solution : $\sqrt{2x-5} < 5$

$2x - 5 \geq 0$ and $\sqrt{2x-5} < 5$

squaring on both side

$2x \geq 5$ $2x - 5 < 25$

$x \geq \dfrac{5}{2}$ $x < 15$

So solution is $x \in \left[\dfrac{5}{2}, 15\right)$

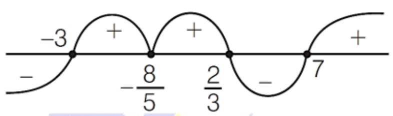
Example: 29 Solve the inequality $(x + 3)(3x - 2)^5 (7 - x)^3 (5x + 8)^2 \geq 0$.

Solution : $(x + 3)(3x - 2)^5 (7 - x)^3 (5x + 8)^2 \geq 0$

$\Rightarrow -(x + 3)(3x - 2)^5 (x - 7)^3 (5x + 8)^2 \geq 0$

$\Rightarrow (x + 3)(3x - 2)^5 (x - 7)^3 (5x + 8)^2 \leq 0$

take before x , + ve sign in all brackets



Hence, $x \in (-\infty, -3] \cup \left[\dfrac{2}{3}, 7\right] \cup \left\{-\dfrac{8}{5}\right\}$.

Example: 30 The solution set of the inequality $\log_{\frac{\sqrt{3}}{2}}(x^2 - 3x + 2) \geq 2$ is

(1) $\left(\dfrac{1}{2}, 2\right)$ (2) $\left(1, \dfrac{5}{2}\right)$

(3) $\left[\dfrac{1}{2}, 1\right) \cup \left(2, \dfrac{5}{2}\right]$ (4) (1,2)

Solution : $\log_{\frac{\sqrt{3}}{2}}(x^2 - 3x + 2) \geq 2$

$\Rightarrow x^2 - 3x + 2 \leq \left(\dfrac{\sqrt{3}}{2}\right)^2$

$\Rightarrow 4x^2 - 12x + 8 \leq 3$

$\Rightarrow 4x^2 - 12x + 5 \leq 0$

$\Rightarrow (2x - 5)(2x - 1) \leq 0$

$\Rightarrow x \in \left[\dfrac{1}{2}, \dfrac{5}{2}\right]$

But domain $x^2 - 3x + 2 > 0$

$\Rightarrow (x - 1)(x - 2) > 0$

$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

Hence $x \in \left[\dfrac{1}{2}, 1\right) \cup \left(2, \dfrac{5}{2}\right]$

Example: 31 If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval

(1) $(2, \infty)$ (2) $(1, 2)$ (3) $(-2, -1)$ (4) $\left(1, \frac{3}{2}\right)$

Solution : $\log_{0.3}(x-1) < \log_{0.09}(x-1)$; $\log_{0.3}(x-1) < \frac{\log_{0.3}(x-1)}{2}$

$\Rightarrow \log_{0.3}(x-1) < 0 \Rightarrow x-1 > 1 \Rightarrow x > 2$

Example: 32 Number of integers satisfying inequality $\log_{\frac{1}{3}}\left(\frac{3x-7}{2x}\right) \geq 0$ is

(1) 6 (2) 7 (3) 4 (4) 5

Solution : $\log_{\frac{1}{3}}\left(\frac{3x-7}{2x}\right) \geq 0$

$0 < \frac{3x-7}{2x} \leq \left(\frac{1}{3}\right)^0$

$\frac{3x-7}{2x} > 0$ and $\frac{3x-7}{2x} \leq 1$

$x \in (-\infty, 0) \cup \left(\frac{7}{3}, \infty\right)$ ..(i) $\frac{x-7}{2x} \leq 0$ $x \in (0, 7]$ ...(ii)

now (i) $\cap$ (ii)

$x \in \left(\frac{7}{3}, 7\right]$

so integer values of x = 3, 4, 5, 6, 7

x = 3, 4, 5, 6, 7

Example: 33 Let $x = (0.15)^{20}$. Find the characterstic and mantissa in the logrithm of x, to the base 10. Assume $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$.

Solution : $\log x = \log(0.15)^{20} = 20\log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2]$

$= 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2]$ $\left[\therefore \log 5 = \log\frac{10}{2}\right]$

$= 20[-1 + \log 3 - \log 2]$

$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824 = -16.48$

Hence characterstic $= -17$ and mantissa $= 0.52$.