Chapter 01

Basic Mathematics

CONTENT

1. ALGEBRA

1.1 Quadratic Equation

An algebraic equation of second order (highest power of the variable is two) is called a quadratic equation. General quadratic equation:-

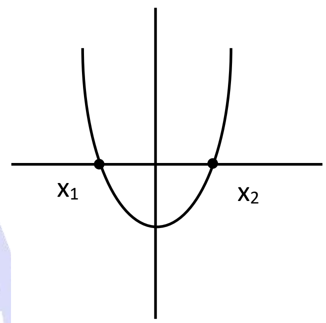
$ax^2 + bx + c = 0$......(i) (where $a \neq 0$).

The general solution of above equation is (roots)

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$



Here x_1 and x_2 are called roots of equation (i).

Then sum of roots = $x_1 + x_2 = -\frac{b}{a}$

and product of roots = $x_1 x_2 = \frac{c}{a}$

Example 1: Find roots of equation $2x^2 - x - 3 = 0$.

(1) 3/2, –1 (2) 1, 3/2 (3) 2, 3 (4) –3/2, –1

Solution: Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$, we have a=2, b=–1, c=–3.

Now from $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)} \Rightarrow x_1 = \frac{6}{4} = \frac{3}{2} \Rightarrow x_2 = \frac{-4}{4} \Rightarrow x_2 = -1$$

Example 2: Find the sum and product of roots of equation $3x^2 - 5x - 12 = 0$.

(1) 5/3, –4 (2) –5/3, 4 (3) –4, 5/3 (4) –4, –5/3

Solution: $\alpha + \beta = -b/a = 5/3 \Rightarrow$ (α, β are roots of equation)

$\Rightarrow \alpha\beta = c/a = -4$

1.2 Binomial Approximation

$\Rightarrow (1+x)^n \approx 1 + nx$ If $x <<< 1$

Example 3: The value of acceleration due to gravity (g) at height h above the surface of earth is given by $g' = g\left(\frac{R_e + h}{R_e}\right)^{-2}$. What is the value of g' for very small height?

Solution: $g' = g\left\{1 + \frac{h}{R_e}\right\}^{-2} \Rightarrow g' = g\left\{1 - \frac{2h}{R_e}\right\}$

Example 4: Calculate the value of $\sqrt{0.99}$

(1) 0.995 (2) 1 (3) 0.990 (4) 1.05

Solution: $\sqrt{0.99} \Rightarrow \left\{1-\frac{1}{100}\right\}^{1/2} \Rightarrow \left\{1-\frac{1}{200}\right\} = 0.995$

1.3 Logarithm formulas:

- $\log m^n = n \log m$
- $\log mn = \log m + \log n$
- $\log\left(\frac{m}{n}\right) = \log m - \log n$
- $\log_a{}^{(a)} = 1$
- $\ell n\, m = \log_e m$

SPOT LIGHT

- $\log_e m = 2.303 \log_{10}(m)$; $\ln(1) = 0$

Example 5: Calculate the value of $\log_{10}12$

(1) 1 (2) 1.05 (3) 1.079 (4) 1.085

Solution: $\log_{10}12 = \log_{10}(4 \times 3) = \log_{10}4 + \log_{10}3 = \log_{10}2^2 + \log_{10}3 = 2\log_{10}2 + \log_{10}3$

$= 2 \times 0.301 + 0.477 = 1.079$

1.4 Application of ratio

- If $\frac{p}{q} = \frac{a}{b}$

  $\Rightarrow \frac{p+q}{p-q} = \frac{a+b}{a-b}$
- If $\frac{p}{q} = \frac{a}{b}$

  $\Rightarrow \frac{p+q}{q} = \frac{a+b}{b}$
- If $\frac{p}{q} = \frac{a}{b}$

  $\Rightarrow \frac{p-q}{q} = \frac{a-b}{b}$

Example 6: Calculate the value of $\frac{a+b}{a-b}$ if $\frac{a}{b} = \frac{5}{2}$?

(1) $\frac{7}{3}$ (2) $\frac{5}{3}$ (3) $\frac{3}{7}$ (4) $\frac{2}{3}$

Solution: $\frac{a}{b} = \frac{5}{2}$ $\Rightarrow \frac{a+b}{a-b} = \frac{5+2}{5-2} = \frac{7}{3}$

1.5 Progression

PROGRESSION

	Arithmetic progression(AP)	Geometrical Progression (GP)	Harmonic Progression(HP)
General form	a, a+d, a+2d, ……a+(n–1)d	a, ar, ar^2,…, ar^{n-1}	$\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d}\cdots\cdots\frac{1}{a+(n-1)d}$
n^{th} term	$a_n = a+(n-1)d$	$a_n = ar^{n-1}$	$a_n = \frac{1}{a+(n-1)d}$
Sum of n terms	$S_n = \frac{n}{2}[a+a+(n-1)d]$ $\Rightarrow \frac{n}{2}[1^{st}\text{ term} + n^{th}\text{ term}]$	$S_n = \frac{a(1-r^n)}{1-r}$ $S_\infty = \frac{1}{1-r}$ sum of ∞ term	
Mean	$A.M. = \frac{a+b}{2}$	$G.M. = (a.b)^{\frac{1}{2}} = \sqrt{ab}$	$H.M. = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$

SPOT LIGHT

➢ arithmetic mean $\geq$ Geometric mean

Example 7: Find sum of first n natural numbers.

(1) $\left[\frac{n(n+2)}{2}\right]$ (2) $\left[\frac{n(n+1)}{2}\right]$ (3) $\left[\frac{n(n+1)}{1}\right]$ (4) $\left[\frac{n(n+2)}{1}\right]$

Solution: Let sum be S_n then

$S_n = 1 + 2 + 3 + \ldots + n$;

$$S_n = \frac{n}{2}[1+n] = \left[\frac{n(n+1)}{2}\right]$$

Example 8 : Calculate A.M., G.M., H.M. of 3, 6

(1) A. M. = 4; G.M. = $3\sqrt{2}$ H.M. = 3.5 (2) A. M. = $3\sqrt{2}$; G.M. = 4.5 H.M. = 4

(3) A. M. = 4.5; G.M. = $3\sqrt{2}$ H.M. = 4 (4) A. M. = $3\sqrt{2}$; G.M. = 5 H.M. = 4

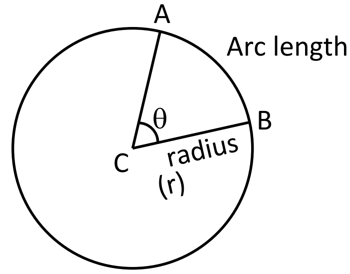
Solution: $A.M. = \frac{a+b}{2} = \frac{3+6}{2} = 4.5$

$G.M. = \sqrt{3\times6} = \sqrt{18} = \sqrt{9\times2} = 3\sqrt{2}$

$H.M. = \frac{2ab}{a+b} = \frac{2\times3\times6}{9} = 4$

2. TRIGONOMETRY

2.1 Degrees and Radian



angle $(\theta) = \dfrac{\text{Arc length}}{\text{Radius}}$ $\Rightarrow$ $\theta = \dfrac{AB}{r}$

If r = 1 then θ = AB (Arc length)

Unit of angle is radian or degree.

Angle Conversion formulas

π radian = 180°

Angle Conversion formulas

Degree to radian

1 degree = $\dfrac{\pi}{180} \approx 0.02\text{rad}$

- Multiply by $\dfrac{\pi}{180}$

Radian to degree

1 radian = $\dfrac{180^\circ}{\pi} \approx 57^\circ$ degree

- Multiply by $\dfrac{180^\circ}{\pi}$

Some Standard values

Degree	30°	45°	60°	90°	120°	135°	150°	180°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	2π

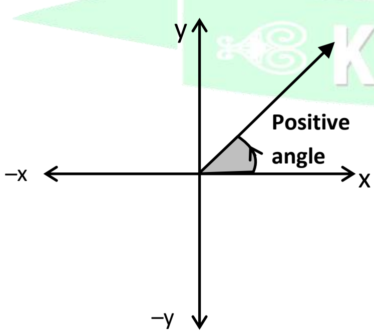
Example 9: (i) Convert $\frac{\pi}{6}$ rad to degrees. (ii) Convert 30° to radians.

Solution: (i) $\dfrac{\pi}{6} \times \dfrac{180^\circ}{\pi} = 30^\circ$ (ii) $30^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi}{6}$ rad

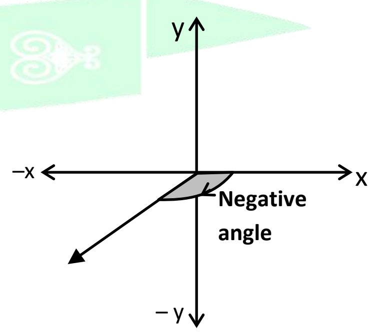
2.2 Measurement of Positive and Negative Angles

Positive angles: - Angle is always positive when measured in anti-clock wise direction

Negative angles: - Angle is always negative when measured in clock wise direction

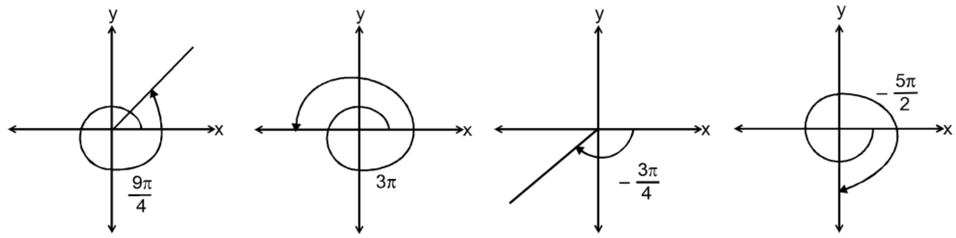


Positive angle of rotation

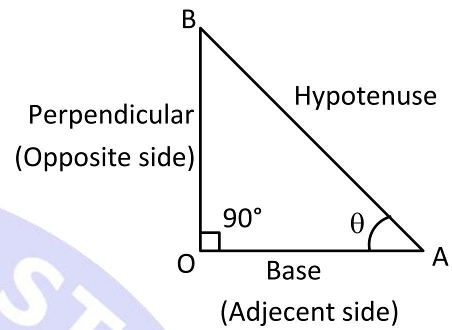


Negative angle of rotation

e.g.: -



2.3 Trigonometric Functions



Sinθ Cosθ tanθ

$$\frac{P \quad B \quad P}{H \quad H \quad B}$$

Cosecθ secθ cotθ

The trigonometric ratio

$\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{OB}{AB}$ $\Rightarrow \text{cosec}\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{OA}{AB}$ $\Rightarrow \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\text{Perpandicular}}{\text{Base}} = \dfrac{OB}{OA}$ $\Rightarrow \cot\theta = \dfrac{1}{\tan\theta}$

➢ **Values of sin θ, cos θ and tan θ for some standard angles.**

Degree	0°	30°	37°	45°	53°	60°	90°
Radian	0	$\frac{\pi}{6}$	$\frac{37\pi}{180}$	$\frac{\pi}{4}$	$\frac{53\pi}{180}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sinθ	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

➢ **Trigonometric ratio of angles greater than 90°**

Step 1 → Identify the quadrant in which angle lies.

↓

Step 2 → Decide +ve or –ve sign by ASTC rule

↓

Step 3 → (i) If angle = $(n\pi \pm \theta)$ where n is an integer. $(0 \le \theta \le \frac{\pi}{2})$

Then trigonometric function of $(n\pi \pm \theta)$ = (sign)same trigonometric function of θ

(ii) If angle = $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$

Then trigonometric function of $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ = (sign)complimentary trigonometric function of θ

($\because$ sign will be decide by ASTC rule)

⇒ **Complimentary trigonometric function**

$\sin\theta \rightleftharpoons \cos\theta$
$\tan\theta \rightleftharpoons \cot\theta$
$\sec\theta \rightleftharpoons \operatorname{cosec}\theta$

➢ **Some common example (For 90° and 270°):-**

(i) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \quad \Rightarrow \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

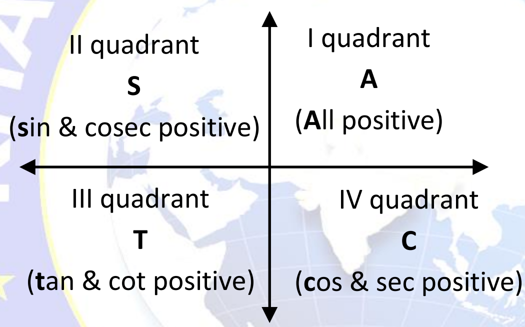
(ii) $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \quad \Rightarrow \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

(iii) $\sec\left(\frac{\pi}{2}-\theta\right) = \operatorname{cosec}\theta \Rightarrow \operatorname{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$

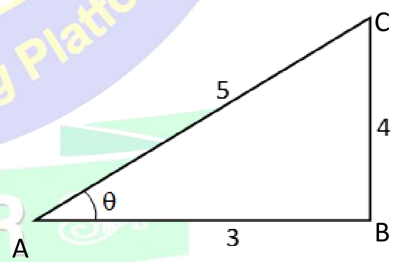
(iv) $\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta \quad \Rightarrow \cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$

SPOT LIGHT

ASTC rule



Example 10: Find the six trigonometric ratios from given figure



Solution:

$\sin\theta = \frac{4}{5}$; acos$\theta = \frac{3}{5}$;

$\tan\theta = \frac{4}{3}$; $\operatorname{cosec}\theta = \frac{5}{4}$;

$\sec\theta = \frac{5}{3}$; $\cot\theta = \frac{3}{4}$

Example 11: Find the value of :

(i) sin30° + cos60°　　(ii) sin 0° – cos 0°

(iii) tan 45°–tan 37°　　(iv) sin 390°

(v) cos 405°　　(vi) tan 420°　　(vii) sin 150°　　(viii) cos 120°

(ix) tan 135° (x) sin (330°) (xi) cos 300° (xii) sin (–30°)

(xiii) cos (–60°) (xiv) tan (–45°) (xv) sin (–150°)

Solution:

(i) $\sin 30° + \cos 60° = \frac{1}{2} + \frac{1}{2} = 1$

(ii) $\sin 0° - \cos 0° = 0 - 1 = -1$

(iii) $\tan 45° - \tan 37° = 1 - \frac{3}{4} = \frac{1}{4}$

(iv) $\sin 390° = \sin(360° + 30°) = \sin 30° = \frac{1}{2}$

(v) $\cos 405° = \cos(360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

(vi) $\tan 420° = \tan(360° + 60°) = \tan 60° = \sqrt{3}$

(vii) $\sin 150° = \sin(90° + 60°) = \cos 60° = \frac{1}{2}$ or $\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$

(viii) $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

(ix) $\tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

(x) $\sin 330° = \sin(360° - 30°) = -\sin 30° = -\frac{1}{2}$

(xi) $\cos 300° = \cos(360° - 60°) = \cos 60° = \frac{1}{2}$

(xii) $\sin(-30°) = -\sin 30° = -\frac{1}{2}$

(xiii) $\cos(-60°) = +\cos 60° = +\frac{1}{2}$

(xiv) $\tan(-45°) = -\tan 45° = -1$

(xv) $\sin(-150°) = -\sin(150°) = -\sin(180° - 30°) = -\sin 30° = -\frac{1}{2}$

2.4 Trigonometric formulae

Trigonometry Formulae

Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

Addition/Subtraction Formulae

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

When A = B

- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  $= 1 - 2\sin^2\theta$
  $= 2\cos^2\theta - 1$

Maximum and Minimum Values

- $-1 \le \sin\theta \le 1$
- $-1 \le \cos\theta \le 1$

- $-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$

DETECTIVE MIND

$a\sin\theta + b\cos\theta = \left(\sqrt{a^2+b^2}\right)\sin(\theta+\alpha)$

Where $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

SPOT LIGHT

- **Small angle approximation**

  if θ is small then $\sin\theta \approx \theta$ (θ in radian)

  $\tan\theta \approx \theta$ (θ in radian) $\sin\theta \approx \tan\theta \approx \theta$

  $\cos\theta \approx 1$

Example 12: Find the approximate values of

(i) sin 1°

(1) 0.02 (2) 0.04 (3) 0.05 (4) 0.03

(ii) tan 2°

(1) 0.04 (2) 0.03 (3) 0.05 (4) 0.01

(iii) cos 1°

(1) 0.7 (2) 1.02 (3) 1.05 (4) 1

Solution: (i) $\sin 1^\circ = \sin\left(1^\circ \times \frac{\pi}{180^\circ}\right) = \sin\frac{\pi}{180} \approx \frac{\pi}{180} = 0.02$

(ii) $\tan 2^\circ = \tan\left(2^\circ \times \frac{\pi}{180^\circ}\right) = \tan\frac{\pi}{90} \approx \frac{\pi}{90} = 0.04$

(iii) $\cos 1^\circ = \cos\left(1^\circ \times \frac{\pi}{180^\circ}\right) = \cos\left(\frac{\pi}{180^\circ}\right) = 1$

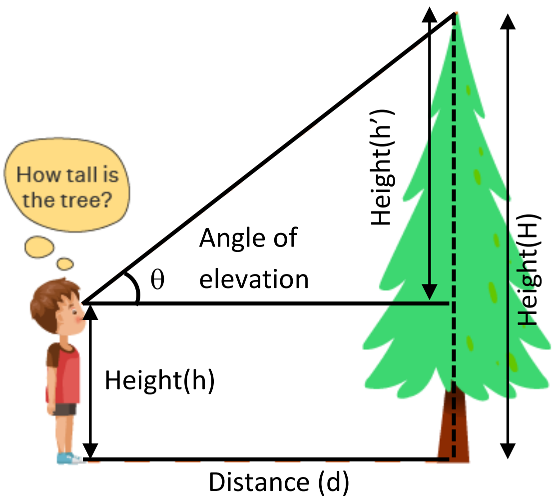
Example 13: A ball is projected with speed u at an angle θ to the horizontal. The range R of the projectile is given by $\frac{u^2\sin 2\theta}{g}$ for which value of θ will the range be maximum for a given speed of projection? (here g = constant)

(1) $\frac{\pi}{2}$ rad (2) $\frac{\pi}{4}$ rad (3) $\frac{\pi}{3}$ rad (4) $\frac{\pi}{6}$ rad

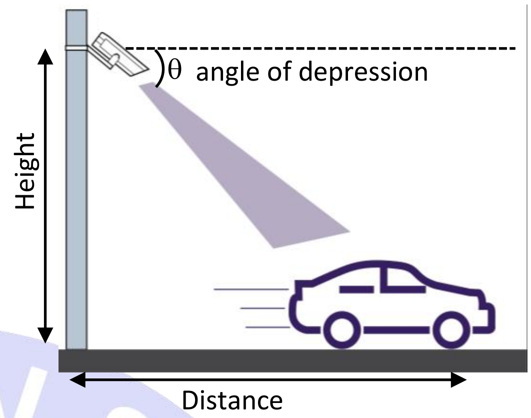
Solution: As $\sin 2\theta \le 1$ so range will be maximum if $\sin 2\theta = 1$. Therefore $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ rad.

- **Application of trigonometry: -**

  (i) Height of object **(ii) Area covered by a camera**

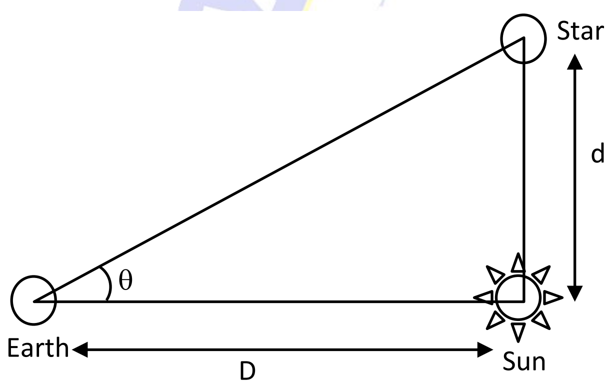


Height (H) = h + distance(d) × tanθ (h')



$$\text{Distance} = \frac{\text{height}}{\tan\theta}$$

(iii) Measuring large distance



$d = D \times \tan\theta$

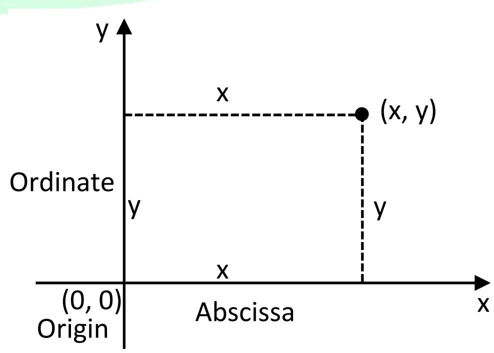
3. COORDINATE GEOMETRY

To specify the position of a point in space, we use right-handed rectangular axes coordinate system. This system consists of origin and axis

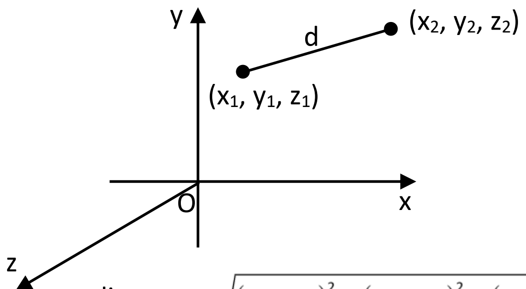
3.1 Position of a point in xy plane :

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure.

Here x-coordinate and y-coordinate is called **Abscissa** and **Ordinate** respectively.

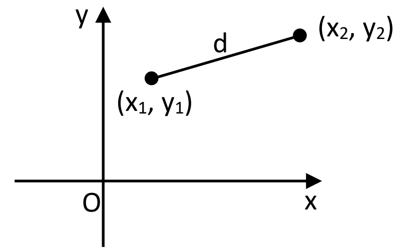


3.2 Distance Formula



distance= $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In xy plane (z= 0)(2D)



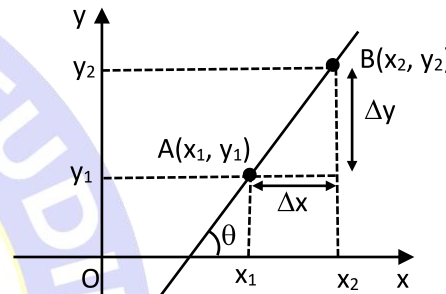
distance= $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

3.3 Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

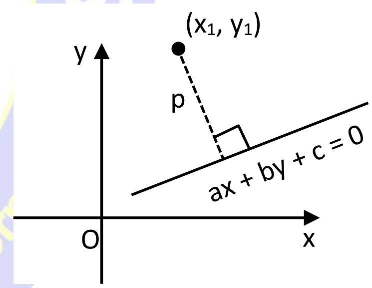
Here θ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.



3.4 Perpendicular distance between a point and a line

The perpendicular distance of a line $ax + by + c = 0$ from a point (x_1, y_1) is

$$\Rightarrow \quad p = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$



Example 14: For point (2, 14) find abscissa and ordinates. Also find distance from y and x-axis.

Solution: Abscissa = x-coordinate = 2 = distance from y-axis.

Ordinate = y-coordinate = 14 = distance from x-axis.

Example 15: Find value of a, if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

(1) – 2 cm or 6 cm (2) – 2 cm or 8 cm

(2) – 3 cm or 8 cm (4) – 4 cm or 6 cm

Solution: By using distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow 13 = \sqrt{[3-(-9)]^2 + [3-a]^2}$$

$\Rightarrow 13^2 = 12^2 + (3–a)^2 \Rightarrow (3–a)^2 = 13^2 – 12^2 = 5^2 \Rightarrow (3–a) = \pm 5 \Rightarrow$ a = –2 cm or 8 cm

Example 16: A dog wants to catch a cat. The dog follows the path whose equation is $y–x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

(1) (2, –2) (2) (2, 2) (3) (–2, 2) (4) (–2, –2)

Solution: Let catching point be (x_1, y_1) then, $y_1–x_1=0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; So possible points are (2, 2) and (–2, –2).

Example 17: A car changes its velocity linearly from 10 m/s to 20 m/s in 5 seconds. Write velocity as a function of time.

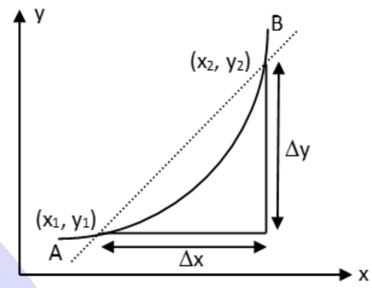
Solution: Slope = $\frac{20-10}{5-0}$ = 2 = m $\Rightarrow$ Line intercept at ν axis = c = 10 $\Rightarrow$ So, equation ν = 2t + 10

4. DIFFERENTIAL CALCULUS (DIFFERENTIATION)

Calculus is the study of relationship between continuously varying functions.

The purpose of differential calculus is to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently.

4.1 Average rate of change :

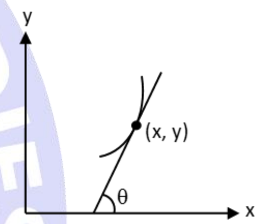


Let a function y = f(x) be plotted as shown in figure. Average rate of change in y w.r.t. x in interval $[x_1, x_2]$ is

Average rate of change = $\frac{\text{change in } y}{\text{change in } x}$

$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ = slope of secant AB.

4.2 Instantaneous rate of change :



It is defined as the rate of change in y with x at a particular value of x(an instant of time). It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x, y) and algebraically by the first derivative of function y =f(x).

Instantaneous rate of change = $\frac{dy}{dx}$ = slope of tangent = $\tan\theta$

4.3 First Derivatives of Commonly used Functions :

y = constant	$\frac{dy}{dx} = 0$
y = cx^n (Power rule)	$\frac{dy}{dx} = cnx^{n-1}$
y = e^x (exponential function)	$\frac{dy}{dx} = e^x$
y = ln x (logarithmic function)	$\frac{dy}{dx} = \frac{1}{x}$
y = sinx	$\frac{dy}{dx} = \cos x$
y = cosx	$\frac{dy}{dx} = -\sin x$
y = tan x	$\frac{dy}{dx} = \sec^2 x$

4.4 Rules For Differentiation

➢ **Sum Rule**

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point, so their derivative $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

The Sum Rule also extends to sums of more than two functions. If $u_1, u_2, ...u_n$ are differentiable $\forall$ x, then

$$\frac{d}{dx}(u_1 + u_2 + ... + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + ... + \frac{du_n}{dx}.$$

➢ **Product Rule**

If u and v are differentiable at x, then The derivative of the product uv is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(\text{I.II}) = \text{I}\frac{d}{dx}(\text{II}) + \text{II}\frac{d}{dx}(\text{I})$$

SPOT LIGHT

While the derivative of the sum of two functions is the sum of their derivatives, but the derivative of the product of two functions is not the product of their derivatives.

e.g. $\frac{d}{dx}(x.x) = \frac{d}{dx}(x^2) = 2x$, while $\frac{d}{dx}(x).\ \frac{d}{dx}(x) = 1.1 = 1.$

- **Quotient Rule**

  If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

  $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

  $\Rightarrow$ the derivative of the quotient of two functions is not the quotient of their derivatives.

- **Function of Functions : Chain Rule**

  Let f be a function of x, which in turn is a function of t. The first derivative of f w.r.t. t is equal to the product of $\frac{df}{dx}$ and $\frac{dx}{dt}$

  $$\therefore\ \frac{df}{dt} = \frac{df}{dx} \times \frac{dx}{dt}$$

- **Power Chain Rule**

  If u(x) is a differentiable function and where n is a Real number , then u^n is differentiable and $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$, $\forall n \in R$

DETECTIVE MIND

- **DIFFERENTIATION OF RADIAN V/S DEGREES**

  $$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°)$$

Example 18: Find derivatives of the following functions:

(i) $y = 2x^3$ (ii) $y = \frac{4}{x}$ (iii) $y = 3e^x$ (iv) $y = 6 \ln x$

(v) $y = \log_{10} x$ (vi) $y = 5 \sin x$

Solution: (i) $y = 2x^3 \Rightarrow \frac{dy}{dx} = 2\,[3x^{3-1}] = 6x^2$ (ii) $y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = 4\,[(-1)x^{-1-1}] = -\frac{4}{x^2}$

(iii) $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$ (iv) $y = 6\ln x \Rightarrow \frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{6}{x}$

(v) $y = \log_{10} x \Rightarrow y = \frac{1}{2.303}\ \ln x \Rightarrow = \frac{dy}{dx} = \frac{1}{2.303x}$ (vi) $y = 5\sin x \Rightarrow \frac{dy}{dx} = 5\cos x$

Example 19: Find first derivative of y w.r.t. x. (i) $y = x^2 \sin x$, (ii) $y = 4(e^x)\cos x$

Solution: (i) $\frac{dy}{dx} = x^2(\cos x) + (2x)(\sin x) = x^2 \cos x + 2x \sin x$

(ii) $\frac{dy}{dx} = 4[(e^x)(\cos x) + (e^x)(-\sin x)] = 4e^x[\cos x - \sin x]$

Example 20: Find dy/dx if y = tan x .

Solution: $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \times \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Example 21: Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$

Solution: We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$\Rightarrow \left(\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\right) \Rightarrow \frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2}$

$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$.

4.5 Double differentiation :

the second derivative of y = f(x) is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 22: (a) $\frac{d}{dx}\cos 3x$, (b) $\frac{d}{dx}\sin 2x$, (c) $\frac{d}{dt}(A \sin(\omega t + \phi)$

Solution: (a) $\frac{d}{dx}\cos 3x = -\sin 3x \frac{d}{dx} 3x = -3 \sin 3x$

(b) $\frac{d}{dx}\sin 2x = \cos 2x \frac{d}{dx}(2x) = \cos 2x . 2 = 2 \cos 2x$

(c) $\frac{d}{dt}(A \sin(\omega t + \phi) = A \cos(\omega t + \phi) \frac{d}{dt}(\omega t + \phi) = A \cos(\omega t + \phi). \omega. = A\omega \cos(\omega t + \phi)$

Example 23: If $y = 4x^2$, what is $\frac{dy}{dt}$

Solution: $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = 8x.\frac{dx}{dt}$

Example 24: If f (x) = x cos x , find f ″ (x).

Solution: Using the Product Rule, we have$\Rightarrow f'(x) = x\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x) = -x \sin x + \cos x$

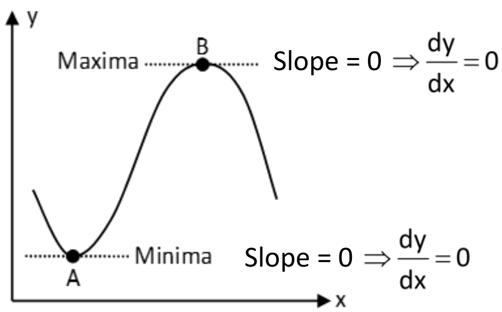
To find f ″ (x) we differentiate $f'(x)$:

$f''(x) = \frac{d}{dx}(-x \sin x + \cos x) = -x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(-x) + \frac{d}{dx}(\cos x) = -x \cos x - \sin x - \sin x$

$= -x \cos x - 2 \sin x$

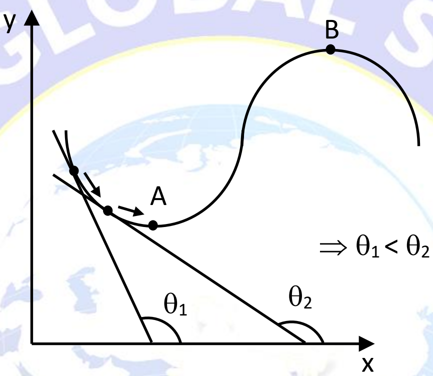
4.6 Maximum and Minimum value

$\Rightarrow$ At the points of maxima and minima, first derivative becomes zero.
Higher order derivatives are used to find the maximum and minimum values of a function.



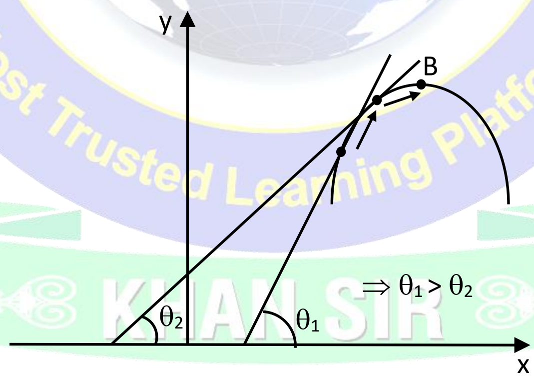
- **At point 'A' (minima) :** As we see in figure, in the neighborhood of A, slope increases

  i.e. $\frac{dy}{dx}$ is $\uparrow$ $\qquad \Rightarrow \frac{d^2y}{dx^2} > 0$.



- **At point 'B' (maxima) :** As we see in figure, in the neighborhood of B, slope decreases

  i.e. $\frac{dy}{dx}$ is $\downarrow$ $\qquad \Rightarrow \frac{d^2y}{dx^2} < 0$.



SPOT LIGHT

Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$ **Condition for maxima :** $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

4.7 Differentiation as a tool

$\frac{dy}{dx}$ is rate of change of 'y' with respect to 'x' :

(i) Velocity :- rate of change of displacement 'x' with respect to time 't' i.e. $\boxed{v =}\frac{dx}{dt}$

(ii) Acceleration :- rate of change of velocity 'v' with respect to time 't' i.e. $\boxed{a =}\frac{dv}{dt}$

(iii) Force:- rate of change of momentum 'p' with respect to time 't' i.e. $\boxed{F =}\frac{dp}{dt}$

Example 25: Find minimum value of

(i) $y = 1 + x^2 - 2x$ (ii) $y = 5x^2 - 2x + 1$.

Solution: (i) $\frac{dy}{dx} = 2x - 2 \Rightarrow$ for minima, $\frac{dy}{dx} = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

$\frac{d^2y}{dx^2} = 2 \Rightarrow \frac{d^2y}{dx^2} > 0$ at x = 1 there is minima for minimum value of y

$y_{minimum} = 1 + 1 - 2 = 0$

(ii) For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive so minima at $x = \frac{1}{5}$.

Therefore $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

Example 26: The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$.

How fast is the area changing with respect to the diameter when the diameter is 10 m?

Solution: The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4}2D = \frac{\pi D}{2}$$

When D = 10 m, the area is changing at rate $(\pi/2)\,10 = 5\pi$ m.

5. INTEGRAL CALCULUS (INTEGRATION)

5.1 Indefinite Integral :

Integration is the reverse process of differentiation. With the help of integration we can find a function whose derivative is known. Consider a function F(x) whose differentiation w.r.t. x is equal to f(x) then

$$\int f(x)dx = F(x) + c$$

here c is the constant of integration and this is called indefinite integration.

➢ **some basic formulae of integration are :**

Indefinite Integral	Reversed derivative formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n = rational number	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ $\frac{d}{dx}(x) = 1$

	$\int dx = \int 1dx = x + C$ (special case)	
2.	$\int \sin x = \cos x + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3.	$\int \cos x = \sin x + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4.	$\int \sec^2 x dx = \tan x + C$	$\frac{d}{dx}(\tan x) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$
6.	$\int \sec x \tan x dx = \sec x + C$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
7.	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$	$\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$
8.	$\int \frac{1}{x} = \ell n(x) + C$	

5.2 RULES FOR INTEGRATION

- **CONSTANT MULTIPLE RULE**

  A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.

  $\int kf(x)dx = k\int f(x)dx$; where k is a constant

- **SUM AND DIFFERENCE RULE**

  A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f an antiderivative of g.

  $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

- **RULE OF SUBSTITUTION**

  $\int f(g(x)) \cdot g'(x)\, dx = \int f(u)du$ 1. Substitute $u = g(x)$, $du = g'(x)\,dx$.

  $= F(u) + C$ 2. Evaluate by finding an antiderivative F (u) of f(u). (any one will do.)

  $= F(g(x)) + C$

  3. Replace u by g(x).

5.3 Definite Integration:

- When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

  Consider a function F(x) whose differentiation w.r.t. x is equal to f(x), in an interval $a \le x \le b$ then

  $$\int_a^b f(x)dx = F(b) - F(a)$$

Example 27: (i) $\int 5x^2 dx$ (ii) $\int \frac{7}{x^2}dx$ (iii) $\int \frac{t}{\sqrt{t}}dx$

Solution: (i) $\frac{5x^3}{3} + C$ (ii) $\int 7x^{-2}dx = \frac{-7}{x} + C = \frac{-7x^{-1}}{1} + C$

(iii) $\int t^{1/2}dt = \frac{t^{3/2}}{3/2} + C = \frac{2}{3}t^{3/2} + C$

Example 28: Integrate the following w.r.t. x.

(i) $4x^3$ (ii) $x-\frac{1}{x}$ (iii) $\frac{1}{2x+3}$ (iv) cos (4x+3)

Solution:

(i) $\int 4x^3dx = 4\left(\frac{x^{3+1}}{3+1}\right)+c = \frac{4x^4}{4}+c = x^4+c$ (ii) $\int\left(x-\frac{1}{x}\right)dx = \int xdx - \int\frac{1}{x}dx = \frac{x^2}{2}-\ln x + c$

(iii) $\int\frac{dx}{2x+3} = \frac{\ln(2x+3)}{2}+c$ (iv) $\int\cos(4x+3)dx = \frac{\sin(4x+3)}{4}+c$

Example 29: Integrate the following w.r.t. x.

(i) $\cos^2x$ (ii) $\sin^2x$

Solution:

(i) $\int\cos^2 x\,dx = \int\frac{2\cos^2 x}{2}\,dx = \int\frac{(1+\cos 2x)}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int\cos 2x\,dx = \frac{x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+c = \frac{x}{2}+\frac{1}{4}\sin 2x + c$

(ii) $\int\sin^2 x\,dx = \int\frac{(1-\cos 2x)}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int\cos 2x\,dx = \frac{x}{2}-\frac{1}{2}\frac{\sin 2x}{2}+c = \frac{x}{2}-\frac{1}{4}\sin 2x + c$

Example 30: The integral $\int_1^5 x^2\,dx$ is equal to

Solution: $\int_1^5 x^2\,dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3}-\frac{1^3}{3}\right] = \frac{125}{3}-\frac{1}{3} = \frac{124}{3}$

Example 31: The integral $\int_{-1}^{4} 3dx$ is equal to

Solution: $3\int_{-1}^{4} dx = 3\,[x]_{-1}^{4} = 3[4-(-1)] = (3)\,(5) = 15$

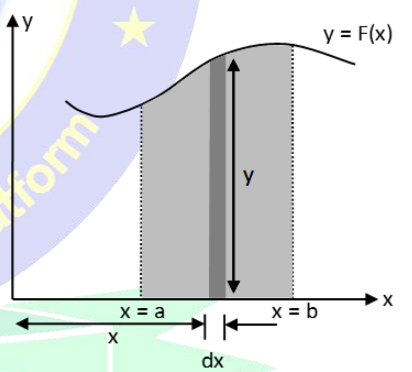
5.4 Application of integration

- **Area under the curve :**

  Area of small shown element = ydx = f(x) dx

  If we sum up all areas between x=a and x= b then

  Area = $\int_a^b f(x)dx$ = (shaded area between curve and x-axis)



- **Average value**

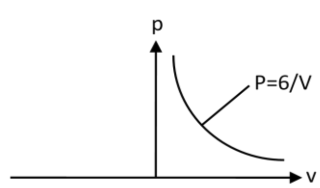
  if y = f (x) then

  $$y_{average} = \frac{\int_{x=a}^{x=b} f(x)dx}{b-a} = \frac{\text{Area under the curve}}{\text{interval}}$$

Example 32: Find area under the curve of y = x from x = 0 to x = a

Solution: $\int_0^a ydx = \left.\frac{x^2}{2}\right|_0^a = \frac{a^2}{2}$

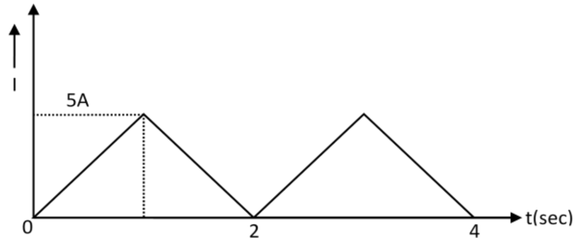
Example 33: A gas expands its volume from V to 3V as shown in figure. Calculate the work in this process if W = $\int pdv$.



(1) 6.6 (2) 6.9 (3) 3 (4) 3.5

Solution: $W = \int_{v}^{3v} p\,dv \Rightarrow \int_{v}^{3v} \frac{6}{v} dv = \left[6\log_e v\right]_{v}^{3v} = 6.6\text{ J}$

Example 34: Calculate average value of current from t = 0 to t = 4 seconds.



(1) 2.5 (2) 3 (3) 3.5 (4) 2

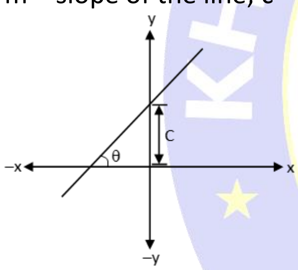
Solution: $I_{average} = \frac{\int_a^b I\,dt}{\int_a^b dt} = \frac{\text{total area}}{\text{total time}} = \frac{5\times 2}{4} = 2.5 \text{ amp}$

6. GRAPHS

6.1 Straight line Equation and its Graph

$\boxed{y = mx + c}$

m = slope of the line; c = intercept on the y-axis.



➢ **Slope:**

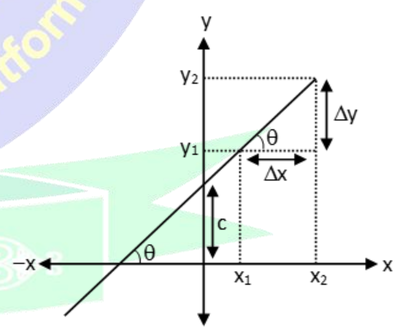
Slope of a line is a quantitative measure to express the inclination of the line. It is expressed by ratio of change in ordinate to change in abscissa.

Slope of a line = slope of tangent = $\frac{dy}{dx}$

When the x and the y axes are scaled identically, slope equals to tangent of the angle, which line makes with the positive x-axis.

$m = \tan\theta$

Sometimes the slope is also called **gradient**.



➢ **Intercept:**

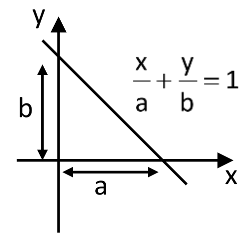
It equals to the value of ordinate y, where the line cuts the y-axis. It may be positive, negative or zero for lines crossing the positive y-axis, negative y-axis and passing through the origin respectively.

DETECTIVE MIND

Intercept form of line:

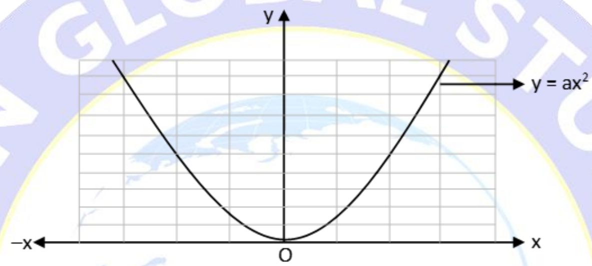
Equation of line in intercept from $\frac{x}{a}+\frac{y}{b}=1$

(where a, b are the intercepts on x-axis and y axis respectively)



6.2 Parabola Equation and its Graph

A function of the form $y = ax^2 + bx + c$ is known as parabola. The simplest parabola has the form $y = ax^2$. Its graph is shown in the following figure.

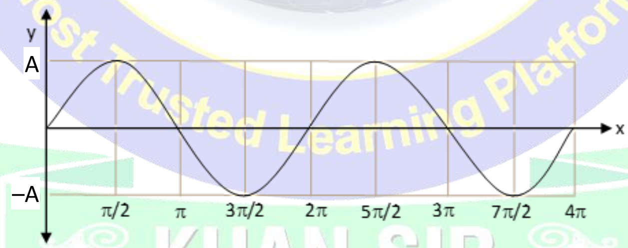


6.3 Trigonometric functions

Among all the trigonometric functions, sinusoidal function, which includes sine and cosine both is most common in use.

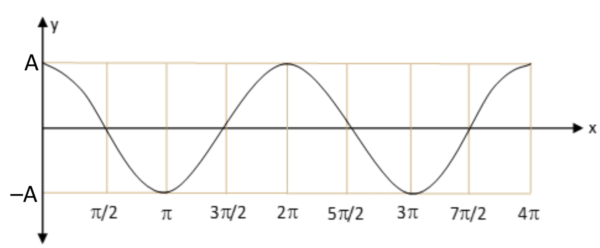
- **Sine Function :** $y = A \sin x$

  Here, A is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a sine function is shown, which has amplitude a units.



- **Cosine Function :** $y = A \cos x$

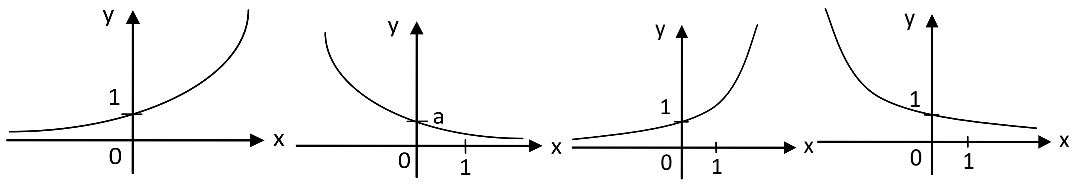
  Here, A is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a cosine function is shown, which has amplitude a units.



6.4 Exponential function and its graph

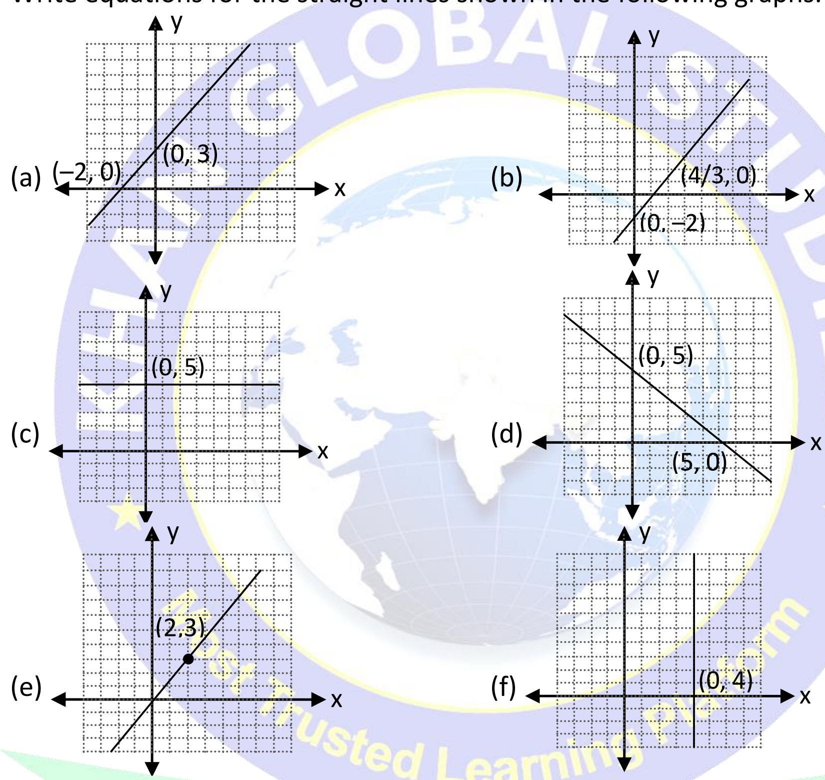
Behaviour of several physical phenomena is described by exponential function to the base e. Here e is known as Euler's Number. e = 2.718218

Most commonly used exponential function has the form $y = ae^{-x}$.



(a) $y = e^x$ (b) $y = ae^{-x}$ (c) $y = 2^x$ (d) $y = (0.5)^x$

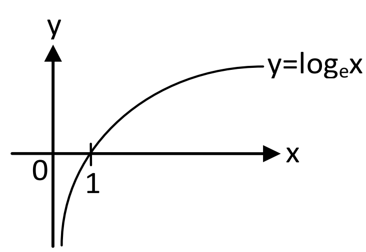
Example 35: Write equations for the straight lines shown in the following graphs.



Solution: (a) $y = \frac{3}{2}x + 3$; (b) $y = \frac{3}{2}x - 2$; (c) $y = 5$;

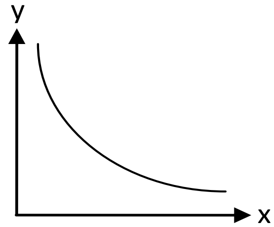
(d) $\frac{y}{5} + \frac{x}{5} = 1$; (e) $y = \frac{3}{2}x$; (f) $x = 4$

6.5 Logarithmic Function



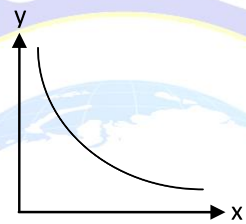
6.6 Rectangular Hyperbola

$xy = c^2$



6.7 Hyperboloids

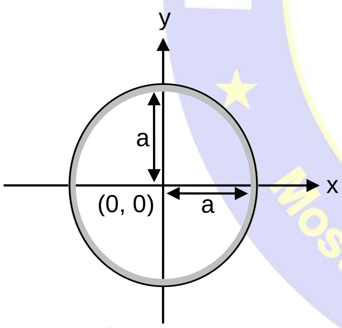
simplest form $y = \frac{k}{x^n}$ (k is a positive constant)

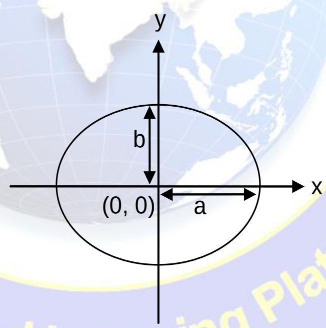


6.8 Circle and Ellipse

Circle : $x^2 + y^2 = a^2$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

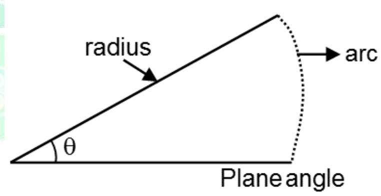




6.9 Plane angle (The angle between two lines)

Plane angle is the radius measurement of rotation of line along a plane.

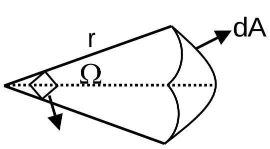
$$\boxed{\text{Plane angle } d\theta = \frac{ds}{r}}$$

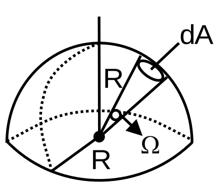


6.10 Solid Angle :

It is the 3D analogue of an angle. Such as that subtended by cone or formed by planes meeting at a point

$$\boxed{\text{Solid angle } \Omega = \frac{dA}{r^2}}$$





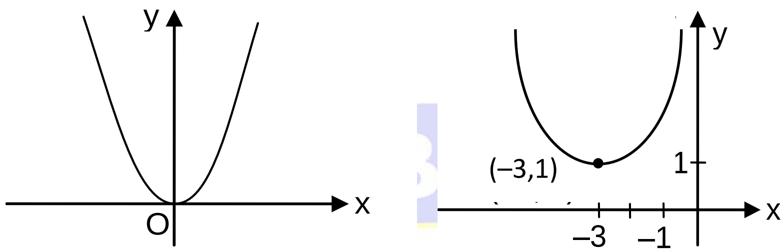
Example 36: Sketch the graph of the function

$f(x) = x^2 + 6x + 10$.

Solution: Completing the square, we write the equation of the graph as

$y = x^2 + 6x + 10 = (x + 3)^2 + 1$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward.



(a) $y = x^2$ (b) $y = (x+3)^2+1$

7. PHYSICAL QUANTITIES

In physics we deal with two type of physical quantity one is scalar and other is vector. Each quantity has magnitude. Magnitude of a physical quantity means product of numerical value and unit of that physical quantity.

PHYSICAL QUANTITIES

Scalar quantity

- Has only magnitude
- It is applicable only for algebra rules
- It can change in term of its magnitude
- It can divide another scalar.
- It is one dimensional quantity.
- It can be added, substracted and Multiplied by basic laws.
- Examples: Mass, Speed, distance etc.

Vector quantity

- Has magnitude & specified direction
- It is applicable only for vector algebraic rules.
- It can change in terms of both (Magnitude & direction)
- It cannot divide other vectors.
- It is multi-dimensional quantity (1D, 2D & 3D)
- It is obeys the law of parallelogram and commutative law of addition.
- Examples: Velocity, acceleration

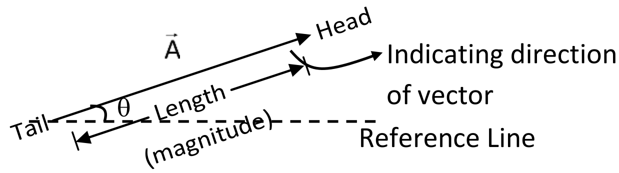
SPOT LIGHT

- If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not a be vector.
  e.g. pressure, surface tension or current etc. have directions but are not vectors.

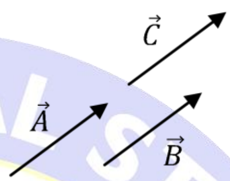
8. VECTOR

8.1 Representation of vector :

- Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as Mathematically, vector is represented by $\vec{A} = |\vec{A}|\hat{A} = A\hat{A}$.



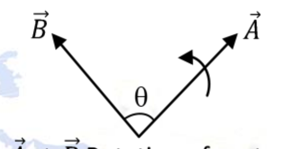
- The magnitude of a vector $(\vec{A})$ is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.
- $\hat{A}$ is the unit vector along the direction of vector $\vec{A}$.
- If a vector is displaced parallel to itself it does not change (see Figure)



$\vec{A} = \vec{B} = \vec{C}$

Transition of a vector parallel to itself

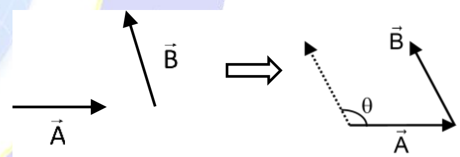
- If a vector is rotated through an angle other than multiple of 2π (or 360°) it changes (see Figure).
- Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.



$\vec{A} \neq \vec{B}$ Rotation of vector

8.2 Angle between two vectors :

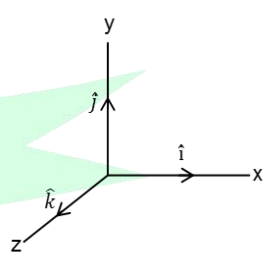
- Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e. $0 \le \theta \le \pi$).



8.3 Type of Vector

- **Unit Vector:**

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector $(\vec{A})$ can be written as the product of unit vector $(\hat{A})$ in that direction and magnitude of the given vector.

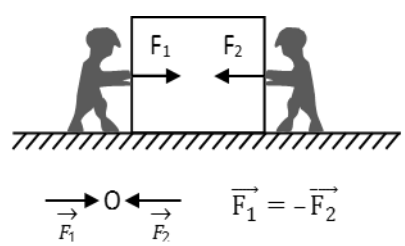


$\vec{A} = A\ \hat{A}$ or $\hat{A} = \dfrac{\vec{A}}{A}$ (Where $\hat{A}$ = unit vector along the direction of vector A)

A unit vector has no dimensions and unit. Unit vectors along the positive x, y and z-axis of a rectangular coordinate system are denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

- **Null (Zero) Vectors :**

By definition, a null(zero) vectors has magnitude "zero" and has an arbitrary direction. This will be clear in the following example:



$\vec{F_1} = -\vec{F_2}$

When two persons push an object with equal and opposite force $\vec{F_1}$ and $\vec{F_2}$, the resultant force is $\vec{F} = \vec{F_1} + \vec{F_2} = 0$ because the objects does not accelerate. Since, the particle's acceleration is zero, but acceleration is vector quantities that means it has direction but **indeterminable**

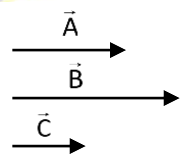
The examples of null vector.

1. Velocity of a stationary object
2. Two people pulling a rope in opposite directions with equal force.
3. Displacement of throwing an object upward and then again holding it at same position.
4. The velocity of a train standing still on a platform.
5. Acceleration of a car going at uniform velocity.

DETECTIVE MIND

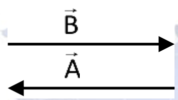
- Zero (null) vector has direction (defined) but **indeterminable.**

- **Parallel vector:-** Those vectors which have same direction are called parallel vectors. Angle between two parallel vector is always 0°.



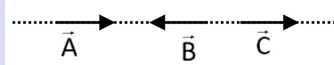
$\vec{A}$, $\vec{B}$ and $\vec{C}$ are parallel.

- **Anti-parallel vectors:-** Vectors which have opposite direction i.e. angle between them is always 180°, are called anti-parallel vector.



- **Collinear Vectors :**

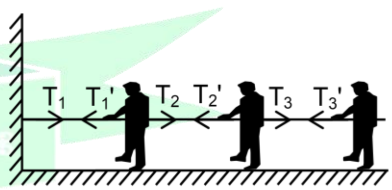
If two or more vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ etc., pass through the same straight line, they are said to be collinear vectors.



$\vec{A},\vec{B},\vec{C}$ act along the same line, i.e. $\vec{A},\vec{B},\vec{C}$ are collinear vector. The collinear vectors $\vec{A},\vec{B},\vec{C}$ need not point in same direction.

For instance, when several persons pull a straight string at its different points, the tensions T_1, T_1', T_2, T_2', ... T_3' etc., in the string are collinear.

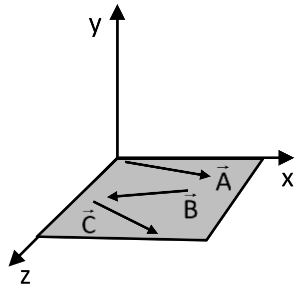
All tensions T_1,T_1', T_2, T_2' etc., are collinear but not unidirectional



- **Coplanar Vectors :**

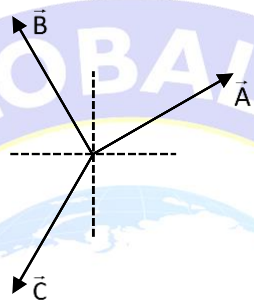
If two or more vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ say lie in the same plane, they are said to be coplanar. Here $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar as they act (lie) in the plane of the paper.

The coplanar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ lie in the plane of the paper.



- **Concurrent Vectors :**

  If line of action of all vectors is passing through same point then vectors are called concurrent vector.

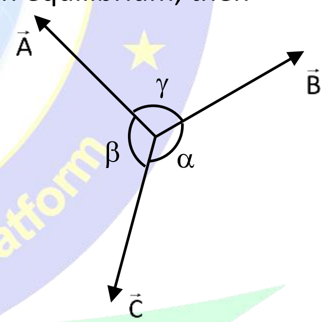


$\vec{A}, \vec{B} \,\&\, \vec{C}$ are concurrent vectors.

DETECTIVE MIND

Lami's theorem: - When three forces acting at a point are in equilibrium, then

$$\Rightarrow \frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$



A, B, C are magnitude of vector $\vec{A}, \vec{B} \,\&\, \vec{C}$ respectively.

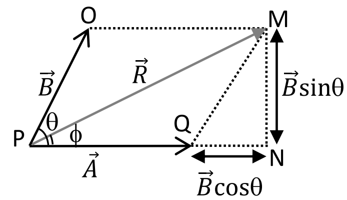
8.4 Multiplication of A Vector by A Scalar

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\left(\vec{B} = \lambda\vec{A}\right)$ whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is λ times $|\vec{A}|$.

If $\vec{B} = \lambda\vec{A} \Rightarrow \left|\vec{B}\right| = \lambda\left|\vec{A}\right|$ and $\hat{B} = \hat{A}$.

8.5 Addition of Vectors

➢ **Parallelogram law :**



If two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram both pointing outwards (and their tails coinciding) as shown. Then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e., vector sum of $\vec{A}$ and $\vec{B}$).

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

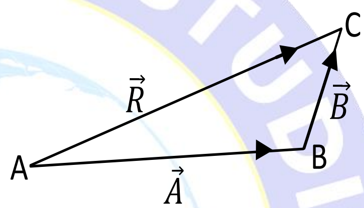
The direction of resultant vector $\vec{R}$ from $\vec{A}$ is given by

$$\tan\phi = \frac{MN}{PN} = \frac{MN}{PQ+QN} = \frac{B\sin\theta}{A+B\cos\theta}$$

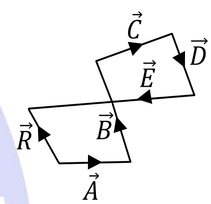
$$\phi = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$$

➢ **Triangle law :**

To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of it at the head of $\vec{A}$. The sum $\vec{A}$ + $\vec{B}$ is a vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A}$ + $\vec{B}$ = $\vec{R}$. As the figure formed is a triangle, this method is called ' triangle method' of addition of vectors. If the 'triangle method' is extended to add any number of vectors in one operation as shown. Then the figure formed is a polygon and hence the name **Polygon Law** of addition of vectors is given to such type of addition.



$\vec{A}+\vec{B}=\vec{R}$



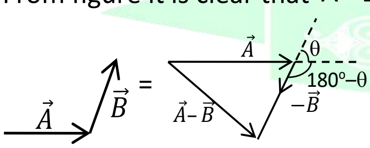
$\vec{R}=\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E}$

8.6 Subtraction of Vectors :

➢ The vector $-\vec{b}$ is vector with the same magnitude as $\vec{b}$ but the opposite direction.
Adding the two vectors.
$\vec{b}+(-\vec{b})=\vec{0}$

➢ Subtraction of a vector from other vector is the addition of negative vector, i.e.,
$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

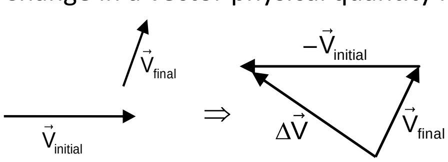
➢ From figure it is clear that $\vec{A}-\vec{B}$ is equal to addition of $\vec{A}$ with reverse of $\vec{B}$



$|\vec{A}-\vec{B}| = [(A)^2 + (B)^2 + 2AB\cos(180° - \theta)]^{1/2}$

$= |\vec{A}-\vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

Change in a vector physical quantity means subtraction of initial vector from the final vector.



- As $R = [A^2 + B^2 + 2AB\cos\theta]^{1/2}$ so R will be maximum when, $\cos\theta$ = max = 1, i.e., $\theta = 0°$, i.e. vectors are like or parallel and $R_{max} = A + B$.
- The resultant will be minimum if, $\cos\theta$ = min = –1, i.e., $\theta = 180°$, i.e. vectors are antiparallel and $R_{min} = |A - B|$
- If the vectors A and B are orthogonal,

  i.e., $\theta = 90°$, $R = \sqrt{A^2 + B^2}$
- Addition of two vector having equal magnitude (A_0)

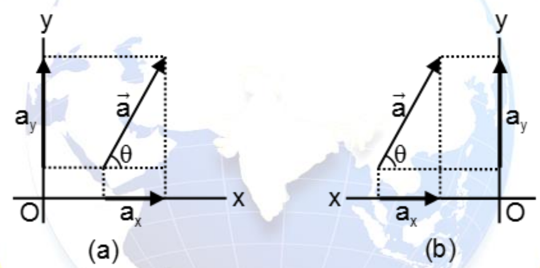
  $R = 2A_0 \cos\frac{\theta}{2}$
- Subtraction of two vector having equal magnitude (A_0)

  $S = 2A_0 \sin\frac{\theta}{2}$

DETECTIVE MIND

- Minimum number of unequal coplanar vectors whose sum can be zero is three.
- The resultant of three non-coplanar vectors can never be zero, or minimum number of non coplanar vectors whose sum can be zero is four.

8.7 Resolution of Vectors

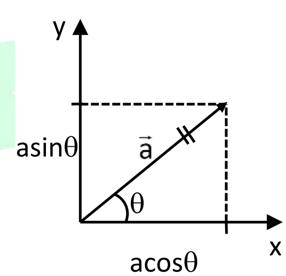


A component of a vector is the projection of the vector on its axis. In this example, a_x is the component of vector on (or along) the x axis and a_y is the component along the y axis. To find the projection of a vector along an axis, we draw perpendicular lines from the two ends of the vector to the axis, as shown. The projection of a vector on an x axis is x component, and similarly the projection on the y axis is the y component. The process of finding the components of a vector is called resolving the vector.

- **Resolution in two dimensions**

  In general, a vector has three components, although for the case of 2D vectors the component along the z axis is zero.

  $\vec{a} = a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j}$



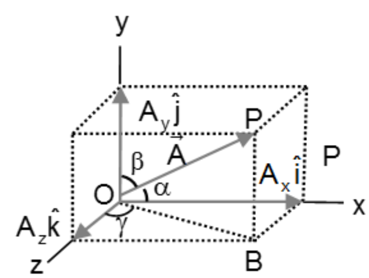
- **Resolution in three dimensions**

  A vector $\vec{A}$ in components along x, y and z-axis can be written as:

  $\Rightarrow \vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = \vec{A}_x\hat{i} + \vec{A}_y\hat{j} + \vec{A}_z\hat{k}$

  $\Rightarrow A = \sqrt{A_x^2 + A_y^2 + A_z^2} \Rightarrow$ $A_x = A\cos\alpha$, $A_y = A\cos\beta$, $A_z = A\cos\gamma$

  where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are termed as **Direction Cosines** of a given vector.



SPOT LIGHT

- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

DETECTIVE MIND

- $\vec{A}$ can be expressed as a sum of two vectors as $\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real numbers)

8.8 Multiplication of Vectors:

- **The scalar product (dot product)**

  $\vec{A}\cdot\vec{B} = AB\cos\theta$ {here θ is the smaller angle between the two vectors when they are arranged with their common initial point}

Properties: -

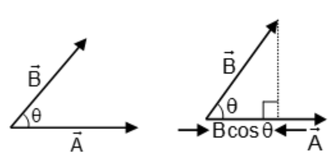
- It is always a scalar which is positive if angle between the vectors is acute (i.e. < 90°) and negative if angle between them is obtuse (i.e. $90° < \theta \le 180°$)
- It is commutative, i.e., $\boxed{\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}}$
- It is distributive, i.e. $\boxed{\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}}$
- The angle between the vectors $\theta = \cos^{-1}\left[\dfrac{\vec{A}\cdot\vec{B}}{AB}\right]$
- Scalar product of two vectors will be maximum when $\cos\theta$ = max = 1, i.e., $\theta = 0°$, i.e., vectors are parallel $\Rightarrow (\vec{A}\cdot\vec{B})_{max} = AB$
- If the scalar product of two nonzero vectors vanishes then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by

  $(\vec{A})^2 = \vec{A}\,.\,\vec{A} = AA\cos\theta = AA\cos 0° = A^2 \quad \Rightarrow \quad A = \sqrt{\vec{A}.\vec{A}}$
- $\hat{n}\cdot\hat{n} = \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
- $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- $\vec{A}\cdot\vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\hat{k})\,.\,(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = [A_xB_x + A_yB_y + A_zB_z]$

- **Projection of Vectors**

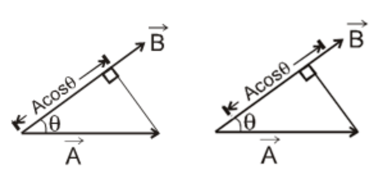
  $\vec{A}\cdot\vec{B} = A\,(B\cos\theta) = B\,(A\cos\theta)$

  Geometrically, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ and $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}\cdot\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.



  Component of $\vec{B}$ along $\vec{A} = B\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{A} = \vec{B}.\hat{A}$

  Component of $\vec{A}$ along $\vec{B} = A\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{B} = \vec{A}.\hat{B}$



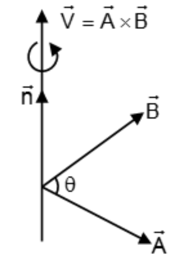
- **Vector Product (cross product) :**

  $\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$

  (here θ is the smaller angle between the two vectors when they are arranged with their common initial point and the direction is given by the right-hand-thumb rule)

  Right-Hand-Thumb Rule:

  To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed, they go towards $\vec{B}$. The direction of the thumb gives the direction of $\hat{n}$.



Properties :

☞ Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

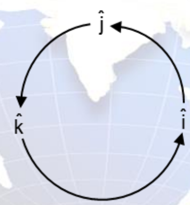
☞ Vector product of two vectors is not commutative i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}|$ = AB sin θ

☞ The vector product is distributive when the order of the vectors is strictly maintained i.e.

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

☞ The magnitude of vector product of two vectors will be maximum when $\sin\theta$ = max = 1, i.e, $\theta = 90°$ $|\vec{A}\times\vec{B}|_{max}$ = AB i.e., magnitude of vector product is maximum if the vectors are orthogonal.

☞ The magnitude of vector product of two non–zero vectors will be minimum when $|\sin\theta|$ = minimum = 0, i.e., θ = 0° or 180° and $|\vec{A}\times\vec{B}|_{min}$ = 0 i.e., if the vector product of two non–zero vectors vanishes, the vectors are collinear.



(B)

☞ $\vec{A}\times\vec{A}$ = AA sin 0° $\hat{n}$ = $\vec{0}$.

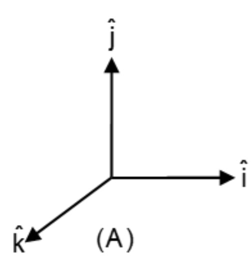
☞ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$

☞ $\hat{i}\times\hat{j}=\hat{k} \quad \hat{j}\times\hat{k}=\hat{i} \quad \hat{k}\times\hat{i}=\hat{j}$

☞ In terms of components,

$$= \vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$



(A)

$\vec{A}\times\vec{B} = \hat{i}\,(A_yB_z - A_zB_y) - \hat{j}\,(A_xB_z - A_zB_x) + \hat{k}\,(A_xB_y - A_yB_x)$

SPOT LIGHT

- The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A}$ and $\vec{B}$ equal to $|\vec{A}\times\vec{B}|$
- If $\vec{A}$ & $\vec{B}$ are the diagonal of a parallelogram then the area will be $\frac{1}{2}|\vec{A}\times\vec{B}|$
- If $\vec{A}$ & $\vec{B}$ are the adjacent sides of a triangle then area of triangle = $\frac{1}{2}|\vec{A}\times\vec{B}|$

SOLVED EXAMPLES

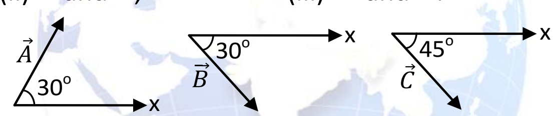
Example 37: Two particles A and B move along the straight lines x+2y +3 = 0 and 2x + y –3 = 0 respectively. Their position vector, at the time of meeting will be ....

(1) $3\hat{i}-3\hat{j}$ (2) $2\hat{i}-2\hat{j}$ (3) $\hat{i}+\hat{j}$ (4) $\hat{i}-\hat{j}$

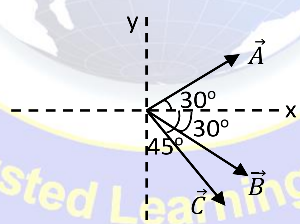
Solution: The particles meet at the point of intersection of lines.
By solving them x=3, y= –3, So position vector of meeting point will be .

Example 38: Three vectors $\vec{A},\vec{B},\vec{C}$ are shown in the figure. Find angle between

(i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.



Solution: To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ such that tails of $\vec{A}.\vec{B}$ and $\vec{C}$ are connected as shown in figure.



Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60°, $\vec{B}$ and $\vec{C}$ is 15° and between $\vec{A}$ and $\vec{C}$ is 75°.

Example 39: A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts West wards. Represent the force in terms of $\hat{i}$.

(1) $10^5\,\hat{i}$ (2) $-10^5\,\hat{i}$ (3) $10^{-5}\,\hat{i}$ (4) $-10^{-5}\,\hat{i}$

Solution: $\vec{F} = -10^5\,\hat{i}$ dynes

Example 40: Find a vector that has the same direction as (–2, 4, 2) but has length 6.

(1) $\sqrt{6}(\hat{i}+2\hat{j}+\hat{k})$ (2) $\sqrt{6}(-\hat{i}+2\hat{j}+\hat{k})$

(3) $\sqrt{6}(\hat{i}-2\hat{j}-\hat{k})$ (4) $\sqrt{6}(-\hat{i}-2\hat{j}+\hat{k})$

Solution: Direction vector is = $\frac{-2\hat{i}+4\hat{j}+2\hat{k}}{2\sqrt{6}}$ then vector is = $6 \times \dfrac{-2\hat{i}+4\hat{j}+2\hat{k}}{2\sqrt{6}}$ = $\sqrt{6}\ (-\hat{i}+2\hat{j}+\hat{k})$

Example 41: A physical quantity (m = 3 kg) is multiplied by a vector $\vec{a}$ such that $\vec{F}=m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a}$ = 3 m/s^2 East wards (ii) $\vec{a}$ = –4 m/s^2 North wards

Solution: (i) $\vec{F}=m\vec{a}$ = 3 × 3 ms^{-2} East wards = 9 N East wards

(ii) $\vec{F}=m\vec{a}$ = 3 × (–4) N North wards = –12N North wards = 12 N South wards

Example 42: Find the sum of vector $\vec{A}=2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B}=4\hat{i}-2\hat{j}-\hat{k}$.

(1) $6\hat{i}-\hat{j}+\hat{k}$ (2) $6\hat{i}+\hat{j}$

(3) $-6\hat{i}-\hat{j}+\hat{k}$ (4) $6\hat{i}+\hat{k}$

Solution: $\vec{A}+\vec{B}=2\hat{i}+3\hat{j}+\hat{k}+4\hat{i}-2\hat{j}-\hat{k}=6\hat{i}+\hat{j}$

Example 43: Find the resultant of two forces each having magnitude F_0, and angle between them is θ.

(1) $2F_0\sin\dfrac{\theta}{2}$ (2) $2F_0\cos\dfrac{\theta}{2}$

(3) $\dfrac{F_0}{2}$ (4) $F_0\cos 2\theta$

Solution: $F^2_{Resultant} = F_0^2 + F_0^2 + 2F_0^2\cos\theta = 2F_0^2(1+\cos\theta) = 2F_0^2\left(1+2\cos^2\dfrac{\theta}{2}-1\right) = 2F_0^2 \times 2\cos^2\dfrac{\theta}{2}$

$F_{resultant} = 2F_0\cos\dfrac{\theta}{2}$

Example 44: On a certain day, rain was falling vertically with a speed of 30 m/sec. A wind started blowing after some time with a speed of $10\sqrt{3}$ m/sec West. Find direction of rain with vertical & resultant speed of rain.

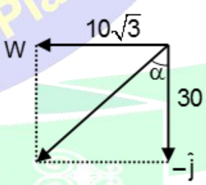
(1) $10\sqrt{3}$ (2) $10\sqrt{2}$ (3) $20\sqrt{3}$ (4) $5\sqrt{3}$

Solution: $\tan\alpha = \dfrac{10\sqrt{3}}{30} = \dfrac{1}{\sqrt{3}}$

α = 30°

Umbrella should be held at an angle of 30°.

Resultant speed of rain = $\sqrt{(10\sqrt{3})^2+30^2} = 20\sqrt{3}$



Example 45: Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$. Find angle between $\vec{A}$ and $\vec{B}$?

Solution: $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}| \Rightarrow A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta \Rightarrow 4AB\cos\theta = 0 \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \dfrac{\pi}{2}$

Example 46: If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference?

(1) $\sqrt{3}$ (2) $\sqrt{2}$ (3) $\sqrt{1}$ (4) $\sqrt{4}$

Solution: Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then

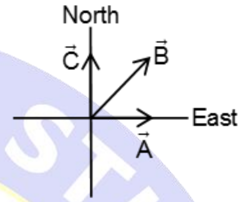
$|\hat{R}| = |\hat{A}+\hat{B}| \Rightarrow 2\cos\dfrac{\theta}{2} = 1$

$$|\vec{R}| = |\hat{A} - \hat{B}| \Rightarrow |\vec{R}| = 2\sin\frac{\theta}{2} = \sqrt{3}$$

Example 47: Vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitude 5, 5 $\sqrt{2}$ and 5 respectively, direction of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are towards East, North-East and North respectively. If $\hat{i}$ and $\hat{j}$ are unit vectors along East and North respectively. Express the sum $\vec{A} + \vec{B} + \vec{C}$ in terms of $\hat{i}$, $\hat{j}$. Also Find magnitude and direction of the resultant.

Solution: $\vec{A} = 5\,\hat{i}$ $\qquad$ $\vec{C} = 5\,\hat{j}$

$$\vec{B} = 5\sqrt{2}\cos 45\,\hat{i} + 5\sqrt{2}\sin 45\,\hat{j} = 5\,\hat{i} + 5\,\hat{j}$$



$$\vec{A} + \vec{B} + \vec{C} = 5\,\hat{i} + 5\,\hat{i} + 5\,\hat{j} + 5\,\hat{j} = 10\,\hat{i} + 10\,\hat{j}$$

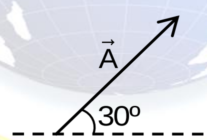
$$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{(10)^2 + (10)^2} = 10\sqrt{2}$$

$$\tan\theta = \frac{10}{10} = 1 \Rightarrow \theta = 45° \text{ from East}$$

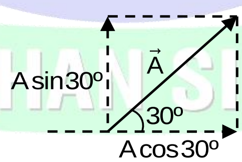
Example 48: A vector makes an angle of 30° with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component?

(1) $\frac{250}{\sqrt{2}}$ $\qquad$ (2) $\frac{250}{\sqrt{4}}$ $\qquad$ (3) $\frac{150}{\sqrt{3}}$ $\qquad$ (4) $\frac{200}{\sqrt{3}}$

Solution: Let vector is $\vec{A}$



$$A_x = A\cos 30° = \frac{A\sqrt{3}}{2} = 250 \Rightarrow A = \frac{500}{\sqrt{3}}$$



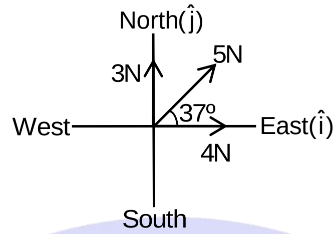
$$A_y = A\sin 30° = \frac{500}{\sqrt{3}} \times \frac{1}{2} = \frac{250}{\sqrt{3}}$$

Example 49: $\vec{A} = \hat{i} + 2\,\hat{j} - 3\,\hat{k}$, when a vector $\vec{B}$ is added to $\vec{A}$, we get a unit vector along x-axis. Find the value of $\vec{B}$? Also find its magnitude.

Solution: $\vec{A} + \vec{B} = \hat{i}$

$\vec{B} = \hat{i} - \vec{A} = \hat{i} - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{j} + 3\hat{k} \Rightarrow |\vec{B}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

Example 50: For given situation, what will be the magnitude of minimum force in newton that can be applied in any direction so that the resultant force is along East direction?



(1) 3N (2) 6N (3) 12N (4) 1.5N

Solution: Let force be F so resultant is in East direction

$4\hat{i} + 3\hat{j} + (5\cos 37°\hat{i} + 5\sin 37°\hat{j}) + \vec{F} = k\hat{i}$

$\Rightarrow 4\hat{i} + 3\hat{j} + 4\hat{i} + 3\hat{j} + \vec{F} = k\hat{i} \Rightarrow 8\hat{i} + 6\hat{j} + \vec{F} = k\hat{i}$

$\Rightarrow \vec{F} = (k-8)\hat{i} - 6\hat{j} \Rightarrow F = \sqrt{(k-8)^2 + (6)^2}$

For F to be minimum, (K – 8) = 0 $\Rightarrow F_{min} = 6$ N

Example 51: Find $\vec{A}\cdot\vec{B}$ if $\vec{A} = 3\hat{i} + 4\hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} + 3\hat{j} - \hat{k}$.

(1) 14 (2) 12 (3) 16 (4) 18

Solution : $\vec{A}\cdot\vec{B} = (3\hat{i} + 4\hat{j} + 2\hat{k}).(2\hat{i} + 3\hat{j} - \hat{k}) = 6 + 12 - 2 = 16$

Example 52: If the Vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?

(1) – 1, 3 (2) 1, 5 (3) – 2, 3 (4) 2, 3

Solution: If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$\Rightarrow \vec{P}\cdot\vec{Q} = 0$ $\Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0$

$\Rightarrow a^2 - 2a - 3 = 0$ $\Rightarrow a^2 - 3a + a - 3 = 0$

$\Rightarrow a(a-3) + 1(a-3) = 0$

$\Rightarrow a = -1, 3$

Example 53: Find angle between $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 12\hat{i} + 5\hat{j}$?

Solution: We have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(3\hat{i}+4\hat{j})\cdot(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$\cos\theta = \frac{36+20}{5\times13} = \frac{56}{65} \Rightarrow \theta = \cos^{-1}\frac{56}{65}$

Example 54: Find $\vec{A}\times\vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 3\hat{i} - \hat{j} + 2\hat{k}$.

Solution: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}\,(-4-(-4)) - \hat{j}\,(2-12) + \hat{k}\,(-1-(-6)) = 10\,\hat{j} + 5\,\hat{k}$

QUICK FOLLOW UP

Mathematical Tools & Vector

Bionomial Approximation

- $(1+x)^n \approx 1 + nx$; If $x <<$

Quadratic eq^n

If $ax^2 + bx + c = 0$......(i)

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $x_1 + x_2 = -\dfrac{b}{a}$
- $x_1 x_2 = \dfrac{c}{a}$

Progression

A.P.

- $a_n = a+(n-1)d$
- $S_n = \dfrac{n}{2}\,[a+a+(n-1)d]$

G.P.

- $a, ar, ar^2, ..., ar^{n-1}$
- $S_n = \dfrac{n}{2}$
- $GM = (ab)^{\frac{1}{2}} = \sqrt{a}$ sum of ∞

Trigonometry

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \mathrm{cosec}^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  $= 1 - 2\sin^2\theta$
  $= 2\cos^2\theta - 1$

Lami's theorm

$$\frac{A}{\sin\alpha} = \frac{\beta}{\sin\beta} = \frac{C}{\sin\gamma}$$

Differentiation

- $\dfrac{d}{dx}\left(cx^n\right) = cnx^{n-1}$
- $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
- $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
- $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{}$

Integration

- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$,
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
- Area $= \int_a^b f(x)dx$

Vector

- $\vec{A} = |A|\,\hat{A}$
- $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$
- $\phi = \tan^{-1}\left(\dfrac{B\sin\theta}{A + B\cos\theta}\right)$
- $\vec{A}\cdot\vec{B} = AB\cos\theta$
- $\theta = \cos^{-1}\left[\dfrac{\vec{A}\cdot\vec{B}}{AB}\right]$
- $\vec{A}\cdot\vec{B} = [A_xB_x + A_yB_y + A_zB_z]$
- $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$
- $\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$
- area of triangle $= \dfrac{1}{2}|\vec{A}\times\vec{B}|$

- **Minima :** $\boxed{\dfrac{dy}{dx} = 0 \text{ and } \dfrac{d^2y}{dx^2} > 0}$
- **Maxima :** $\boxed{\dfrac{dy}{dx} = 0 \text{ and } \dfrac{d^2y}{dx^2} > 0}$