Chapter 03

Quadratic Equation

CONTENT

1.INTRODUCTION:

11 Polynomial :

An expression of the type $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_0$, where $a_0, a_1, a_2 \ldots a_n$ are constants is called a polynomial of degree 'n', where $n \in N$ and a_n which is called **leading coefficient** of the polynomial should not be equal to zero.

1.2 Quadratic Expression:

A second degree expression in one variable which contains the variable with an exponent of 2; but not higher power is called a quadratic expression.

$\Rightarrow$ Ex. $y = ax^2 + bx + c$,

where a = leading coefficient & c = absolute term of quadratic polynomial.

1.3 QUADRATIC EQUATION:

If quadratic polynomial is equated to zero, the equation formed is called a quadratic equation.

Ex. : $ax^2 + bx + c = 0;\ a \neq 0$

$\Rightarrow$ If leading coefficient is 1 then polynomial is called **monic polynomial**. Solving a quadratic equation means finding the values of x for which $ax^2 + bx + c$ vanishes. These values of x are called the roots of quadratic equation.

2. SOLUTION OF QUADRATIC EQUATION :

2.1 Factorization Method :

Let $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation. Hence α and β are the solutions of quadratic equation.

2.2 Quadratic Formula (Sri Dharacharya Method) :

$ax^2 + bx + c = 0.$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let the roots of the given quadratic equation are α & β.

Then $\alpha = \frac{-b + \sqrt{D}}{2a}$ and $\beta = \frac{-b - \sqrt{D}}{2a}$, where $D = b^2 - 4ac$

3. SUM AND PRODUCT OF ROOTS

$ax^2 + bx + c = 0$; $a \neq 0$;

If α, β are the roots then $\alpha + \beta = \frac{-b}{a}$; $\alpha\beta = \frac{c}{a}$ and $\alpha - \beta = \pm \frac{\sqrt{D}}{a}$

4. FORMATION OF A QUADRATIC EQUATION WHEN ROOTS ARE GIVEN :

Let α and β be the given roots of a quadratic equation, then

$(x - \alpha)(x - \beta) = 0$

$x^2 - x(\alpha + \beta) + \alpha\beta = 0$

x^2 – x (sum of the roots) + (Product of the roots) = 0

DETECTIVE MIND

Some Transformation in terms of $\alpha + \beta$ and $\alpha\beta$ ($\alpha, \beta \in R$) :

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2$

5. FORMATION OF QUADRATIC EQUATION WHOSE ROOTS ARE SYMMETRIC EXPRESSION OF α AND β:

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ then finding another quadratic equation whose roots are $2\alpha + 3$, $2\beta + 3$.

Suppose $2\alpha + 3 = y \Rightarrow \alpha = \frac{y-3}{2}$

Put the value of α in the given equation ($\because$ α is its roots) and get a quadratic in y.

$$\frac{a(y-3)^2}{4} + \frac{b(y-3)}{2} + c = 0$$

$$a(y - 3)^2 + 2b(y - 3) + 4c = 0$$

$$ay^2 + 2y(b - 3a) + 9a - 6b + 4c = 0$$

Replace y by x and get the desired equation.

$$ax^2 + 2x(b - 3a) + 9a - 6b + 4c = 0.$$

DETECTIVE MIND

If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta$ $\Rightarrow ax^2 - bx + c = 0$ (Replace x by – x)

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}$ $\Rightarrow cx^2 + bx + a = 0$ $\left(\text{Replace x by } \frac{1}{x}\right)$

(iii) $k\alpha, k\beta$ $\Rightarrow ax^2 + kbx + k^2c = 0.$ $\left(\text{Replace x by } \frac{x}{k}\right)$

(iv) $k + \alpha, k + \beta$ $\Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (Replace x by (x – k))

(v) $\frac{\alpha}{k}, \frac{\beta}{k}$ $\Rightarrow k^2ax^2 + kbx + c = 0$ (Replace x by kx)

6. NATURE OF ROOTS :

Consider the quadratic equation

$ax^2 + bx + c = 0$, and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the **discriminant** of the quadratic equation which decides nature of the roots.

Consider the equation, $ax^2 + bx + c = 0$ having α, β as its roots and $D = b^2 - 4ac$

- $D = 0$
  Roots are equal
  $\alpha = \beta = -b/2a$
- $D \neq 0$
  Roots are unequal
  - $a, b, c \in R$ & $D > 0$
    Roots are real and distinct
    - $a, b, c \in Q$ & D is perfect square
      Roots are rational
      - $a = 1$, $b, c \in I$ & D is perfect square of an integer
        Roots are integral.
    - $a, b, c \in Q$ & D is not perfect square
      Roots are irrational and in conjugate pairs i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$
    Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

SOLVED EXAMPLES

Example: 1 If one root of the equation $4x^2 + 2x - 1 = 0$ is α, then other root is

(1) 2α (2) $4\alpha^3 - 3\alpha$ (3) $4\alpha^3 + 3\alpha$ (4) None of these

Solution: Let α and β are roots of the given equation, then

$$\alpha + \beta = -\frac{1}{2} \Rightarrow \beta = -\frac{1}{2} - \alpha$$

Now $4\alpha^2 + 2\alpha - 1 = 0 \Rightarrow 4\alpha^2 = 1 - 2\alpha$ ..... (1)

Now $4\alpha^3 = \alpha - 2\alpha^2 = \alpha - \frac{1}{2}(1 - 2\alpha)$ [from (1)]

$$\therefore 4\alpha^3 - 3\alpha = -2\alpha - \frac{1}{2}(1 - 2\alpha) = -\frac{1}{2} - \alpha = \beta$$

Example: 2 If $x^2 - 4x + \log_{1/2} a = 0$ does not have two distinct real roots, then maximum value of a is

(1) $\frac{1}{4}$ (2) $\frac{1}{16}$ (3) $\frac{-1}{4}$ (4) None of these

Solution: Since $x^2 - 4x + \log_{1/2} a = 0$ does not have two distinct real roots, discriminant ≤ 0

$$\Rightarrow 16 - 4\log_{1/2} a \leq 0 \Rightarrow \log_{1/2} a \geq 4 \Rightarrow a \leq \frac{1}{16}, \therefore \text{ max. } a = \frac{1}{16}$$

Example: 3 A quadratic polynomial p(x) has $1+\sqrt{5}$ and $1-\sqrt{5}$ roots and it satisfies p (1) = 2. Find the quadratic polynomial.

Solution: Sum of the roots = 2, product of the roots = - 4

$\therefore$ Let $p(x) = a(x^2 - 2x - 4)$

$\Rightarrow$ $p(1) = 2 \Rightarrow 2 = a(1^2 - 2.1 - 4)$ $\Rightarrow$ $a = -\frac{2}{5}$

$\therefore$ $p(x) = -\frac{2}{5}(x^2 - 2x - 4)$

Example: 4 If the equation $(k - 2)x^2 - (k - 4)x - 2 = 0$ has difference of roots as 3 then the value of k is

(1) 1,3 (2) $3, \frac{3}{2}$ (3) $2, \frac{3}{2}$ (4) $\frac{3}{2}, 1$

Solution: Now, $\alpha + \beta = \left(\frac{k-4}{k-2}\right), \alpha\beta = \frac{-2}{k-2}$

$$\therefore (\alpha - \beta) = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}} = \frac{\sqrt{k^2 + 16 - 8k + 8(k-2)}}{k-2}$$

$$3 = \frac{\sqrt{k^2 + 16 - 8k + 8(k-2)}}{k-2}$$

$3k - 6 = \pm k$

$k = 3, \frac{3}{2}$

Example: 5 For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Solution: Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$........(i)

Let D be the discriminant of equation (i).

Roots of equation (i) will be equal if D = 0.

or $4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

or $4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

or $m^2 - 3m = 0$

or $m(m - 3) = 0$

$\therefore$ m = 0, 3.

Example: 6 Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution: Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow$ $D = (a - 10)^2 - 4$ $\Rightarrow$ $4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and D = 0.

$\Rightarrow$ $(a - 10) = \pm 2$ $\Rightarrow$ a = 12, 8

Example: 7 If the roots of the quadratic equation $x^2 - 4x - \log_3 a = 0$ are real, then the least value of a is-
(1) 81 (2) 1/81
(3) 1/64 (4) None of these

Solution: Since the roots of the given equation are real.
$\therefore \quad D \geq 0 \Rightarrow 16 + 4 \log_3 a \geq 0$
$\Rightarrow \quad \log_3 a \geq -4 \Rightarrow a \geq 3^{-4} \Rightarrow a \geq 1/81$
Hence, the least value of a is 1/81.

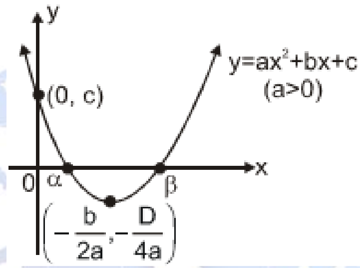
Example: 8 The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be-
(1) $x^2 + 4x - 1 = 0$ (2) $x^2 - 4x - 1 = 0$ (3) $x^2 + 4x + 1 = 0$ (4) None of these

Solution: Given root = $\frac{1}{2+\sqrt{5}} = \sqrt{5} - 2$

So the other root = $-\sqrt{5} - 2$. Then sum of the roots = – 4, product of the roots = – 1
Hence the equation is $x^2 + 4x - 1 = 0$

7. QUADRATIC EXPRESSION AND ITS GRAPH:

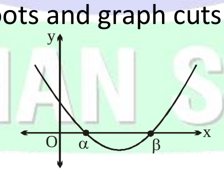
In $y = ax^2 + bx + c$, if a, b, c $\in$ R and a $\neq$ 0. Graph of quadratic takes the shape of a parabola.



1. The graph between x , y is always a parabola.
2. If a > 0 then the shape of the parabola is concave upwards & if a < 0 then the shape of the parabola is concave downwards.
3. The co–ordinate of vertex is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$
4. The parabola intersect the y–axis at point (0, c)
5. The x–coordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation f(x) = 0. Hence the parabola may or may not intersect the x–axis at real points.

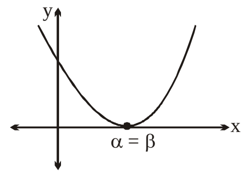
Note:
For different values of a, b, c if graph $y = ax^2 + bx + c$ is plotted then following 6 different shapes are obtained.

Case-I : If a > 0 and D > 0
Then quadratic equation has two roots and graph cuts the x-axis at two distinct points.



(i) y > 0, if x < α or x > β
(ii) y < 0, if α < x < β

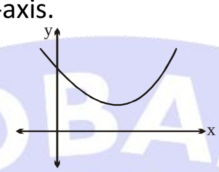
Case-II : If a > 0 and D = 0
Then curve touches x-axis. Hence both zeroes of polynomial coincides.



In this type, equation becomes $y = a(x - \alpha)^2$ and $y \geq 0$, for all $x \in R$.

Case-III : If $a > 0$ and $D < 0$

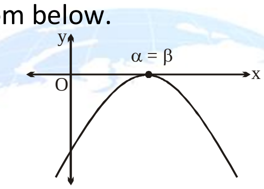
Then curve completely lies above x-axis.



In this case imaginary roots appears and $y > 0$ for all $x \in R$.

Case-IV : If $a < 0$ and $D = 0$

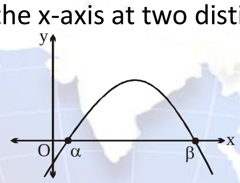
Then graph touches the x-axis from below.



In this case $x \in R$, $y \leq 0$ for all $x \in R$.

Case-V : If $a < 0$ and $D > 0$

Then graph is downward and cuts the x-axis at two distinct points.



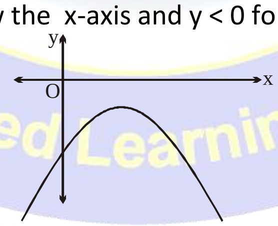
In this case

(i) $y > 0$, if $\alpha < x < \beta$

(ii) $y < 0$, if $x < \alpha$ or $x > \beta$

Case-VI : If $a < 0$ and $D < 0$

Then graph lies completely below the x-axis and $y < 0$ for all $x \in R$.



DETECTIVE MIND

- $\forall x \in R, y > 0$ if an only if $a > 0$ and $b^2 - 4ac < 0$
- $\forall x \in R, y < 0$ if and only if $a < 0$ and $b^2 - 4ac < 0$

8. THEORY OF EQUATIONS :

Relation between roots and coefficients of polynomial equation:

If $\alpha_1, \alpha_2, \alpha_3 ... \alpha_n$ are the roots of the equation;

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

Where $a_0, a_1, ... a_n$ are all real and $a_0 \neq 0$ then,

$$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \Sigma\alpha_1\alpha_2 = +\frac{a_2}{a_0},$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, ..., \sum\alpha_1\alpha_2\alpha_3...\alpha_n = (-1)^n \frac{a_n}{a_0},$$

Note:

1. If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

2. Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

3. If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

4. If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & D is not a perfect square

5. If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must has atleast one real root between 'a' and 'b'.

6. Every equation f(x) = 0 of odd degree has atleast one real root of a sign opposite to that of constant term.

7. A polynomial equation of degree odd with real coefficient must have at least one real root as imaginary roots always occur in pair of conjugates.

9. CONDITION OF COMMON ROOTS:

9.1 Condition for one common root:

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root α.

Hence $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$\therefore \quad \alpha = \dfrac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$, which is the required condition.

This is also the condition that the two quadratic functions $a_1x^2 + b_1xy + c_1y^2$ and $a_2x^2 + b_2xy + c_2y^2$ may have a common factor.

9.2 Condition for both the common roots:

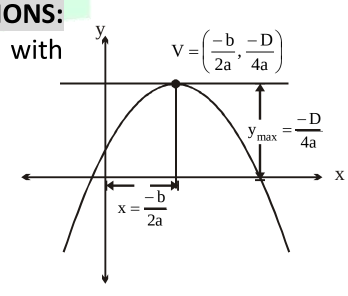
If both roots of the given equations are common then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

DETECTIVE MIND

If the roots of the equation $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio then $\dfrac{b_1^2}{b_2^2} = \dfrac{a_1c_1}{a_2c_2}$

10. MAXIMUM AND MINIMUM VALUES OF QUADRATIC AND RATIONAL FUNCTIONS:

(i) $y = ax^2 + bx + c$ attains its maximum value or minimum value at the point with abscissa $x = \dfrac{-b}{2a}$ according to $a < 0$ or $a > 0$.



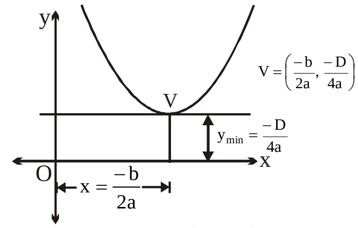
Now, If $a < 0$, then $y_{max} = \dfrac{-D}{4a}$ and it occurs at $x = \dfrac{-b}{2a}$.

If a > 0, then $y_{min} = \frac{-D}{4a}$ and it occurs at $x = \frac{-b}{2a}$. where $D = b^2 - 4ac$

Range when $x \in R$:

If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

If $a < 0 \Rightarrow f(x) \in \left(-\infty, -\frac{D}{4a}\right]$



(ii) Range of functions expressed in the form of $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are either linear or quadratic polynomials.

TYPE-1 : $y = \frac{ax+b}{px+q} \quad \frac{(\text{Linear})}{(\text{Linear})}$.

Example: Find the range of the function. $y = \frac{3x+2}{x-1}, x \neq 1$

Solution: $y = \frac{3x-3+5}{x-1} \Rightarrow y = 3 + \frac{5}{x-1} \Rightarrow y - 3 = \frac{5}{x-1} \Rightarrow x - 1 = \frac{5}{y-3}$

$\Rightarrow x = \frac{5}{y-3} + 1 = \frac{5+y-3}{y-3} \Rightarrow x = \frac{y+2}{y-3}, y \neq 3$

for y = 3, x is not defined

$\therefore$ range is R – {3}

TYPE-2 $y = \frac{ax+b}{px^2+qx+r} \quad \frac{(\text{linear})}{(\text{quadratic})}$

Example: If x is real then find the range of the function $y = \frac{x+2}{x^2+3x+6}$

Solution: $y = \frac{x+2}{x^2+3x+6}$

$\Rightarrow x^2y + 3xy + 6y = x + 2 \Rightarrow x^2y + x(3y-1) + 6y - 2 = 0$

$\because$ x is real $\therefore D \geq 0$

$\Rightarrow (3y-1)^2 - 4y(6y-2) \geq 0 \Rightarrow 9y^2 - 6y + 1 - 24y^2 + 8y \geq 0$

$\Rightarrow 15y^2 - 2y - 1 \leq 0 \Rightarrow y \in \left[-\frac{1}{5}, \frac{1}{3}\right]$

TYPE-3 : $y = \frac{ax^2+bx+c}{px^2+qx+r} \left(\frac{\text{Quadratic}}{\text{Quadratic}}\right)$

Example: If x is real then prove that $y = \frac{x^2-3x+4}{x^2+3x+4}$ lies from $\frac{1}{7}$ to 7.

Solution: $\frac{x^2-3x+4}{x^2+3x+4} = y$

$x^2(y-1) + 3x(y+1) + 4(y-1) = 0$ ... (i)

For a quadratic equation, co-efficient of $x^2 \neq 0$

$\therefore y \neq 1$

$\because$ x is real $\therefore D \geq 0$

$\Rightarrow 9(y+1)^2 - 16(y-1)^2 \geq 0 \Rightarrow -7y^2 + 50y - 7 \geq 0$

$\Rightarrow 7y^2 - 50y + 7 \leq 0$

$\Rightarrow$ $(7y-1)(y-7) \le 0$ $\Rightarrow$ $y \in \left[\frac{1}{7}, 7\right]$ but y = 1 is not included.

If y = 1 $\Rightarrow \frac{x^2-3x+4}{x^2+3x+4} = 1$ $\Rightarrow$ 6x = 0 $\Rightarrow$ x = 0

$\therefore$ y = 1 is also one of the value in the range .

Hence, $y \in \left[\frac{1}{7}, 7\right]$

SOLVED EXAMPLES

Example: 9 The value of 'p' for which the sum of the square of the roots of $2x^2-2(p-2)x-p-1=0$ is least, is

(1) 1 (2) $\frac{3}{2}$ (3) 2 (4) -1

Solution: Let α and β be two roots of the equation

$2x^2-2(p-2)x-p-1=0$. Then

$\alpha+\beta=p-2$ and $\alpha\beta=\frac{-p-1}{2}$

Let $S=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta = (p-2)^2+p+1=p^2-3p+5=\left(p-\frac{3}{2}\right)^2+\frac{11}{4}$

Which is least when $p=\frac{3}{2}$

Example: 10 The value of m for which one of the roots of $x^2-3x+2m=0$ is double of one of the roots of $x^2-x+m=0$ is

(1) 0, 2 (2) $0,-2$ (3) $2,-2$ (4) None of these

Solution: Let α be the root of $x^2-x+m=0$ and 2α be the root of $x^2-3x+2m=0$. Then, $\alpha^2-\alpha+m=0$ and $4\alpha^2-6\alpha+2m=0$

$$\frac{\alpha^2}{-m+3m}=\frac{\alpha}{2m-m}=\frac{1}{-3+2} \Rightarrow \frac{\alpha^2}{2m} = \frac{\alpha}{m}=-1$$

$\Rightarrow m^2=-2m$

$\Rightarrow m=0, m=-2$

Example: 11 The value of the expression $x^2+2bx+c$ will be positive if

(1) $b^2-4c>0$ (2) $b^2-4c<0$ (3) $c^2<b$ (4) $b^2<c$

Solution: Expression $=(x+b)^2-b^2+c = (x+b)^2+(c-b^2)$

$\therefore$ expression will be positive if $c-b^2>0 \Rightarrow b^2<c$

Example: 12 If the roots of the equations $x^2+3x+2=0$ and $x^2-x+\lambda=0$ are in the same ratio then the value of λ is given by

(1) $\frac{2}{7}$ (2) $\frac{2}{9}$ (3) $\frac{9}{2}$ (4) $\frac{7}{2}$

Solution: If roots are in same ratio then $\frac{b_1^2}{b_2^2}=\frac{a_1c_1}{a_2c_2}$

$$\frac{3^2}{(-1)^2}=\frac{(1).(2)}{(1).(\lambda)}\Rightarrow 9=\frac{(2)}{(\lambda)}\Rightarrow \lambda=\frac{2}{9}$$

Example: 13 Let $p(x)=0$ be a polynomial equation of least possible degree, with rational coefficients, having $\sqrt[3]{7}+\sqrt[3]{49}$ as one of its roots. Then the product of all the roots of $p(x)=0$ is

(1) 7 (2) 49 (3) 56 (4) 63

Solution: $x=\sqrt[3]{7}+\sqrt[3]{49}$

$\Rightarrow x^3=7+49+3\sqrt[3]{7}.\sqrt[3]{49}\left(\sqrt[3]{7}+\sqrt[3]{49}\right)\Rightarrow x^3-21x-56=0\Rightarrow$ Product of root = 56.

Example: 14 The equations $x^2+b^2=1-2bx$ and $x^2+a^2=1-2ax$ have one and only one root common. Then

(1) $a-b=2$ (2) $a-b+2=0$ (3) $|a-b|=2$ (4) All of these

Solution: $(x+b)^2=1\Rightarrow x=-b\pm1$

$(x+a)^2=1\Rightarrow x=-a\pm1$

Clearly, $a\neq b$. So, one and only one root common means, $-b+1=-a-1$ or $-b-1=-a+1$

$\Rightarrow$ a – b = –2 or a – b = 2

$\Rightarrow$ $|a-b|=2$

Example: 15 Let P(x) = ax^2 + bx + 8 is a quadratic polynomial. If the minimum value of P(x) is 6 when x = 2, find the values of a and b.

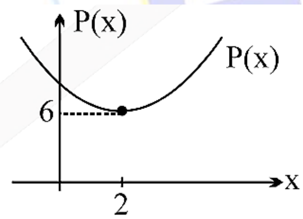
Solution: P(x) = ax^2 + bx + 8 … (1)

P(2) = 4a + 2b + 8 = 6 … (2)

$-\frac{b}{2a}=2;$ $\therefore 4a=-b$

From (2),we get – b + 2b = – 2 $\Rightarrow$ b = – 2

$\therefore$ 4a = – (– 2) $\Rightarrow$ a = 1/2



Example: 16 If the expression $x^2-11x+a$ and $x^2-14x+2a$ have a common factor and $a\neq0$, then the common factor is

(1) $(x-3)$ (2) $(x-6)$ (3) $(x-8)$ (4) None of these

Solution: Here Let $x-\alpha$ is the common factor

then $x=\alpha$ is root of the corresponding equation

$\therefore \alpha^2-11\alpha+a=0$

$\alpha^2-14\alpha+2a=0$

Subtracting $3\alpha-a=0\Rightarrow\alpha=\frac{a}{3}$

Hence $\frac{a^2}{9}-11\frac{a}{3}+a=0, a=0$ or $a=24$

since $a\neq0, a=24$

the common factor of $\begin{cases}x^2-11x+24\\x^2-14x+48\end{cases}$ is clearly (x – 8)

Example: 17 If $Q_1(x) = x^2 + (k - 29)x - k$ and $Q_2(x) = 2x^2 + (2k - 43)x + k$ both are factors of a cubic polynomial P(x), then the largest value of k is

(1) 0 (2) 33 (3) 23 (4) 30

Solution: Two quadratic polynomials can be a factor of cubic polynomial only when they have atleast one root common

$\Rightarrow x^2 + (k-29)x - k = 0$ …… (1)

and $2x^2 + (2k-43)x + k = 0$ ...... (2)

Must have a common root

Multiplying equation (1) by 2 and subtracting, we get

$15x+3k = 0 \Rightarrow x = \frac{-k}{5}$ is the common root

and it must satisfy equation (1)

$\Rightarrow \frac{k^2}{25} + (k-29)\left(-\frac{k}{5}\right) - k = 0 \quad \Rightarrow \left(-\frac{k}{5}\right)\left[-\frac{k}{5}+k-29+5\right]=0$

$\Rightarrow k = 0$ or $k = 30$

Example: 18 Find the maximum value of $f(x) = -3x^2 + 6x + 5$.

Solution: Since, $a < 0$

$\therefore \quad f(x)\max = -\frac{D}{4a} = -\left(\frac{36-4(-3)5}{4(-3)}\right) = \frac{36+60}{12} = 8$

Alternative method:

$f(x) = -3(x^2 - 2x + 1) + 5 + 3 = -3(x - 1)^2 + 8$. Clearly $f(x)_{max} = 8$ at $x = 1$.

Example: 19 If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, show that its other root is a root of $x^2 + (2a - m)x + a^2 - am + n = 0$.

Solution: Let α be a root of the equation $x^2 + ax + b = 0$ which is also a root of $x^2 + mx + n = 0$.

Let β be the other root of $x^2 + ax + b = 0$, then $\alpha + \beta = -a$.

We have $\alpha = -a - \beta$

Since α is a root of $x^2 + mx + n = 0$, we get

$(-a-\beta)^2 + m(-a-\beta) + n = 0$

or $\beta^2 + 2a\beta + a^2 - ma - m\beta + n = 0$

or $\beta^2 + (2a - m)\beta + a^2 - ma + n = 0$

Thus, β is a root of $x^2 + (2a - m)x + a^2 - ma + n = 0$.

Example: 20 If α, β are roots of the equation $ax^2 + 3x + 2 = 0$ $(a < 0)$, then $\frac{\alpha^2}{\beta} + \frac{\beta}{\alpha}$ is greater than-

(1) 0 (2) 1 (3) 2 (4) None of these

Solution: Since $a < 0$, therefore discriminant

$D = 9 - 8a > 0$. So, α and β are real.

We have $\alpha + \beta = \frac{-3}{a}$ and $\alpha\beta = \frac{2}{a}$

$\therefore \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{(\alpha+\beta)^3}{\alpha\beta} - 3(\alpha+\beta)$

$= -\frac{27}{2a^2} + \frac{9}{a} < 0 \quad [\because a < 0]$

So, option (4) is correct.

11. RESOLVING A GENERAL QUADRATIC EXPRESSION IN x AND y INTO TWO LINEAR FACTORS:

The condition that a quadratic function $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

OR

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

12. SOLUTION OF QUADRATIC INEQUALITIES:

The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0$ $(a \neq 0)$ $a, b, c \in R$ are:

1. If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots, $\alpha < \beta$.

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

$a < 0 \Rightarrow x \in (\alpha, \beta)$

2. If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.

Then $a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$

$a < 0 \Rightarrow x \in \phi$

3. If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.

Then $a > 0 \Rightarrow x \in R$

$a < 0 \Rightarrow x \in \phi$

13. LOCATION OF ROOTS :

Let $f(x) = ax^2 + bx + c$, where $a > 0$ and $a, b, c \in R$.

1. Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $b^2 - 4ac \geq 0; f(d) > 0$ and $(-b/2a) > d$.

2. Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'd' (in other words the number 'd' lies between the roots of $f(x) = 0$) is $f(d) < 0$ and $b^2 - 4ac > 0$.

3. Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) and other root is neither d nor e. i.e. $d < x < e$ are $b^2 - 4ac > 0$ and $f(d) \cdot f(e) < 0$.

4. Conditions that both roots of $f(x) = 0$ to lie between the numbers p and q are $(p < q) \cdot b^2 - 4ac \geq 0; f(p) > 0; f(q) > 0$ and $p < (-b/2a) < q$.

5. Condition that one root is greater than e and the other root is less than d are: $b^2 - 4ac > 0$; $f(d) < 0$ and $f(e) < 0$

Extra Result:

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$, $a, b, c \in R$ is :

A quadratic equation if	$a \neq 0$	Two Roots
A linear equation if	$a = 0, b \neq 0$	One Root
A contradiction if	$a = b = 0, c \neq 0$	No Root
An identity if	$a = b = c = 0$	Infinite Roots

SOLVED EXAMPLES

Example: 21 Prove that the expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorized into two linear factors. Find them.

Solution: $2x^2 + 3xy + y^2 + 2y + 3x + 1$

Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we get

$a = 2, h = \frac{3}{2}, b = 1, g = \frac{3}{2}, f = 1, c = 1$

Clearly, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ Given expression can be factorized into two linear factor

To find the factors, form the quadratic equation in x

$2x^2 + 3x(y+1) + (y+1)^2$

$\Rightarrow x = \frac{-3(y+1) \pm \sqrt{(y+1)^2}}{4}$ $\Rightarrow 4x = -3(y+1) \pm (y+1)$

$\Rightarrow 4x = -2(y+1)$ or $4x = -4(y+1)$

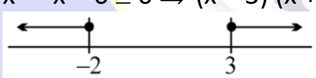
$\Rightarrow 2x + y + 1 = 0$ or $x + y + 1 = 0$

Example: 22 If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.

Solution: $4y^2 + 4xy + x + 6 = 0, y \in R$

$\therefore D \geq 0 \Rightarrow 16x^2 - 16(x+6) \geq 0$

$\Rightarrow x^2 - x - 6 \geq 0 \Rightarrow (x-3)(x+2) \geq 0$



$\therefore x \in (-\infty, -2] \cup [3, \infty)$

Example: 23 Find the values of 'x' for which the inequality $-1 \leq \frac{3x^2 - 7x + 8}{x^2 + 1} \leq 1$, is satisfied.

Solution: $-1 \leq \frac{3x^2 - 7x + 8}{x^2 + 1} \leq 1$

$\Rightarrow \frac{3x^2 - 7x + 8}{x^2 + 1} \geq -1$ and $\frac{3x^2 - 7x + 8}{x^2 + 1} \leq 1$

$\Rightarrow 4x^2 - 7x + 9 \geq 0$ and $2x^2 - 7x + 7 \leq 0$

$\Rightarrow x \in R$ and $x \in \phi$

No value of x is present which satisfy the given inequality.

Example: 24 $(x^2 + 3x + 1)(x^2 + 3x - 3) \geq 5$

Solution: $(x^2 + 3x + 1)(x^2 + 3x - 3) \geq 5$

Let $x^2 + 3x = t$

$\Rightarrow (t+1)(t-3) \geq 5 \Rightarrow t^2 - 2t - 8 \geq 0$

$\Rightarrow (t-4)(t+2) \geq 0$

$\Rightarrow\ t \in (-\infty, -2] \cup [4, \infty)$

$\Rightarrow\ x^2 + 3x \le -2$ or $x^2 + 3x \ge 4$

$\Rightarrow\ x^2 + 3x + 2 \le 0$ or $x^2 + 3x \ge 4$

$\Rightarrow\ (x + 1)(x + 2) \le 0$ or $(x + 4)(x - 1) \ge 0$

$\Rightarrow\ x \in [-2, -1]$ or $x \in (-\infty, -4] \cup [1, \infty)$

Taking union

$x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$

Example: 25 If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then

(1) $a < 1$ (2) $a > \frac{11}{9}$ (3) $a > \frac{3}{2}$ (4) $a < \frac{5}{2}$

Solution: The quadratic equation $f(x) = x^2 - 6ax + 2 - 2a + 9a^2 = 0$ ...(1)

will have real roots if $D = 36a^2 - 4(2 - 2a + 9a^2) \ge 0$

$\Rightarrow\ -8(1 - a) \ge 0$ or $a \ge 1$ ...(2)

The roots of (1) will exceed 3 if

$\frac{-b}{2a} = -\left(\frac{-6a}{2}\right) = 3a > 3$ or $a > 1$ ...(3)

and $f(3) = 9 - 18a + 2 - 2a + 9a^2 > 0$

$\Rightarrow\ 9a^2 - 20a + 11 > 0$

$\Rightarrow (9a - 11)(a - 1) > 0$

$\Rightarrow \left(a - \frac{11}{9}\right)(a - 1) > 0$

$\Rightarrow\ a < 1$ or $a > \frac{11}{9}$ ...(4)

Thus (2), (3) and (4) will hold simultaneously if $a > \frac{11}{9}$.

Example: 26 Find the value of k for which one root of the equation of $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2.

Solution: $x^2 - (k + 1)x + k^2 + k - 8 = 0$

$a > 0$, hence $f(2) < 0$ $\Rightarrow$ $4 - (k + 1)2 + k^2 + k - 8 < 0$

$k^2 - k - 6 < 0$ $\Rightarrow$ $(k - 3)(k + 2) < 0$

$k \in (-2, 3)$

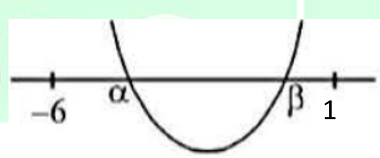
Example: 27 If α, β are the roots of the quadratic equation

$x^2 + 2(k-3)x + 9 = 0$ $(\alpha \ne \beta)$. If $\alpha, \beta \in (-6, 1)$ then find the values of k.

Solution: $x^2 + 2(k - 3)x + 9 = 0$

$\alpha, \beta \in (-6, 1)$

Since leading coefficient is 1, hence



(i) $D \ge 0$

$4(k - 3)^2 - 4 \times 9 \ge 0$

$(k - 6)k \ge 0$

$k \in (-\infty, 0] \cup [6, \infty)$

(ii) $f(-6) > 0$

$36 - 12(k - 3) + 9 > 0$

$36 + 36 - 12k + 9 > 0$

$12k < 81$

$k < \frac{27}{4}$

(iii) $f(1) > 0$

$1 + 2(k - 3) + 9 > 0$

$1 + 2k - 6 + 9 > 0$

$2k + 4 > 0$

$k > -2$

(iv) $-6 < \frac{-b}{2a} < 1$

$-6 < 3 - k < 1$

$2 < k < 9$

Taking intersection of above four condition we get $k \in \left[6, \frac{27}{4}\right)$

Example: 28 Find all the values of k for which one root of the quadratic equation $(k-5)x^2 - 2kx + k - 4 = 0$ is smaller than 1 and the other root exceed 2.

Solution: $(k - 5)x^2 - 2kx + k - 4 = 0$

Case I

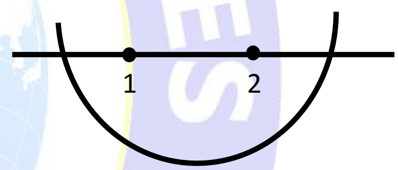
(i) $k - 5 > 0$

$k > 5$

(ii) $f(1) < 0$

$(k - 5) - 2k + k - 4 < 0$

$-9 < 0$



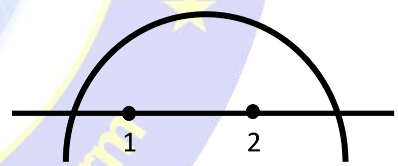
(iii) $f(2) < 0$

$4(k - 5) - 4k + k - 4 < 0$

$k - 24 < 0$

$k < 24$

$k \in (5, 24)$



Case II:

(i) $k - 5 < 0$

$k < 5$

(ii) $f(1) > 0$

$-9 > 0$ that is not possible.

Hence solution is $k \in (5, 24)$.