KGS IAS

Maths Optional

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sequences

$\{f_n\}$ or (f_n) or $\langle f_n \rangle$

→ $\langle a_n \rangle$ or $\{a_n\}$ or (a_n).

Ex: $\langle a_n \rangle$, $a_n = \frac{1}{n}$

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ...

$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{---} \rangle$

or $\langle \frac{1}{n} \rangle$

A sequence is a function from the set of natural number to the set of reals.

$$f : \mathbb{N} \to \mathbb{R}$$

$$f(n)$$

$f(1)$, $f(2)$, $f(3)$, $f(4)$, - - -

↓

f_1, f_2, f_3, f_4, - - -

Ex: $\langle a_n \rangle$ $a_1 = 2$

$$a_{n+1} = \frac{a_n + 2}{3a_n}$$

$$a_2 = \frac{a_1 + 2}{3a_1} = \frac{2+2}{3 \times 2} = \frac{2}{3}$$

$$a_3 = \frac{a_2 + 2}{3a_2} = \frac{\frac{2}{3} + 2}{3 \times \frac{2}{3}}$$

$$= \frac{\frac{8}{3}}{2} = \frac{4}{3}$$

$\langle 2, \frac{2}{3}, \frac{4}{3}, \ldots \rangle$

Ex: $\langle (-1)^n \rangle$ $a_n = (-1)^n$

$a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $a_4 = 1$ ......

$\langle -1, 1, -1, 1, -1, 1, \ldots \rangle$

Ex: $\langle n^2 \rangle$

$\langle 1^2, 2^2, 3^2, 4^2, \ldots \rangle$

Ex $\langle 1+(-1)^n \rangle$

Range $= \{0, 2\}$.

Ex: $\langle n^2 \rangle$

$\langle 1^2, 2^2, 3^2, \dots \rangle$

Range $= \{1^2, 2^2, 3^2, \dots\}$

Range of a sequence

$\langle a_n \rangle$

Range $= \{a_n : n \in \mathbb{N}\}$

Ex: $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$

Range $= \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Ex: $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$

Range $= \{-1, 1\}$.

Sum, Difference, product, Quotient of sequences

$X \equiv \langle x_n \rangle$, $Y \equiv \langle y_n \rangle$

$X \pm Y \equiv \langle x_n \pm y_n \rangle$

$X \cdot Y \equiv \langle x_n \cdot y_n \rangle$

$\dfrac{X}{Y} \equiv \left\langle \dfrac{x_n}{y_n} \right\rangle$, $y_n \neq 0$ $\forall n$.

Constant seq:

$\langle a_n \rangle$

$a_n = \alpha \quad \forall n \in \mathbb{N}$.

$\langle \alpha, \alpha, \alpha, - \;—\rangle$

$\text{Range} = \{\alpha\}$,

→ A sequence $\langle a_n \rangle$ is said to be bounded iff it is bounded above as well as below.

i.e. ∃ real numbers k and K s.t.

$$k \le a_n \le K \quad \forall n \in \mathbb{N}.$$

Bounds of a sequence

→ A sequence $\langle a_n \rangle$ is said to be bounded above iff ∃ some real no K s.t.

$$a_n \le K \quad \forall n \in \mathbb{N}.$$

→ A sequence $\langle a_n \rangle$ is said to be bounded below iff ∃ some real no. k s.t.

$$k \le a_n \quad \forall n \in \mathbb{N}.$$

(iii) $\langle (-1)^n \rangle$ — bounded

$$-2 < (-1)^n < 2 \quad \forall n \in \mathbb{N}$$

Bdd.

(iv) $\langle -n^2 \rangle$

$$\equiv \langle -1^2, -2^2, -3^2, - - \rangle$$

$$-n^2 \leq -1 \quad \forall n \in \mathbb{N}$$

Bdd above.

Ex: (i) $\langle \frac{1}{n} \rangle$ — bdd.

$$0 < \frac{1}{n} \leq 1 \quad \forall n \in \mathbb{N}$$

(ii) $\langle n^2 \rangle$

$$\equiv \langle 1^2, 2^2, 3^2, 4^2, \cdot - \rangle$$

$$1 \leq n^2 \quad \forall n \in \mathbb{N}.$$

Bounded below.

But not bdd above.

Result: A seq. $\langle a_n \rangle$ is bounded if and only if $\exists$ a +ve real number M s.t. $|a_n| \leq M \quad \forall n$.

Proof: Suppose $\langle x_n \rangle$ is bdd.

$\therefore \exists$ real numbers k and K s.t.

$$k \leq a_n \leq K \quad \text{--- (1)}$$

$\forall n \in \mathbb{N}$.

Note: (1) A sequence is bounded iff its range is bdd.

(ii) A sequence is unbounded iff it is not bdd.

Ex (1) $\langle -n^2 \rangle$

(ii) $\langle n^2 \rangle$

(iii) $\langle (-1)^n \cdot n \rangle$

<u>Conversely</u> Suppose that

$\exists$ a +ve real number M

s.t.

$|a_n| \leq M \quad \forall n.$

$\Rightarrow -M \leq a_n \leq M \quad \forall n$

$\therefore \langle a_n \rangle$ is bdd.

Let $M = \max\{|k|, |K|\}$

$|k| \leq M \quad , \quad |K| \leq M$

$\Rightarrow -M \leq k \leq M, \; -M \leq K \leq M$ (III)

(I) & (II) $\Rightarrow$

$-M \leq k \leq a_n \leq K \leq M$

$\Rightarrow -M \leq a_n \leq M \qquad \forall n \in \mathbb{N}$

$\forall n \in \mathbb{N}.$

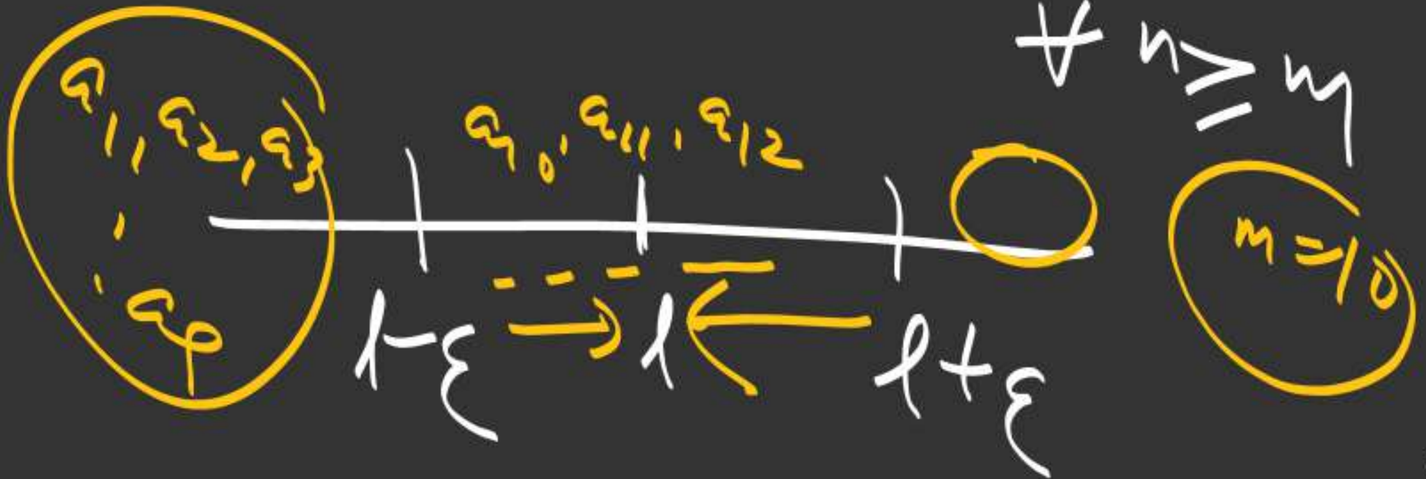
$\Rightarrow |a_n| \leq M \quad \forall n \in \mathbb{N}$

$$\lim_{n\to\infty} a_n = l$$

For each $\varepsilon > 0$, $\exists$ a +ve int m (depending on ε) s.t.

$$|a_n - l| < \varepsilon, \quad \forall n \geq m$$

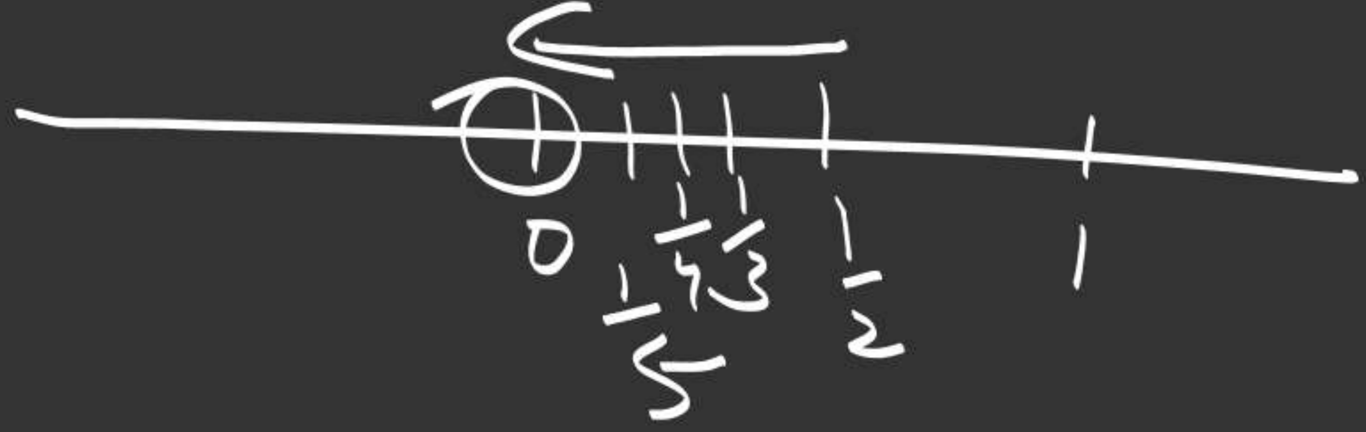
$$\Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon, \quad \forall n \geq m$$



Ex:

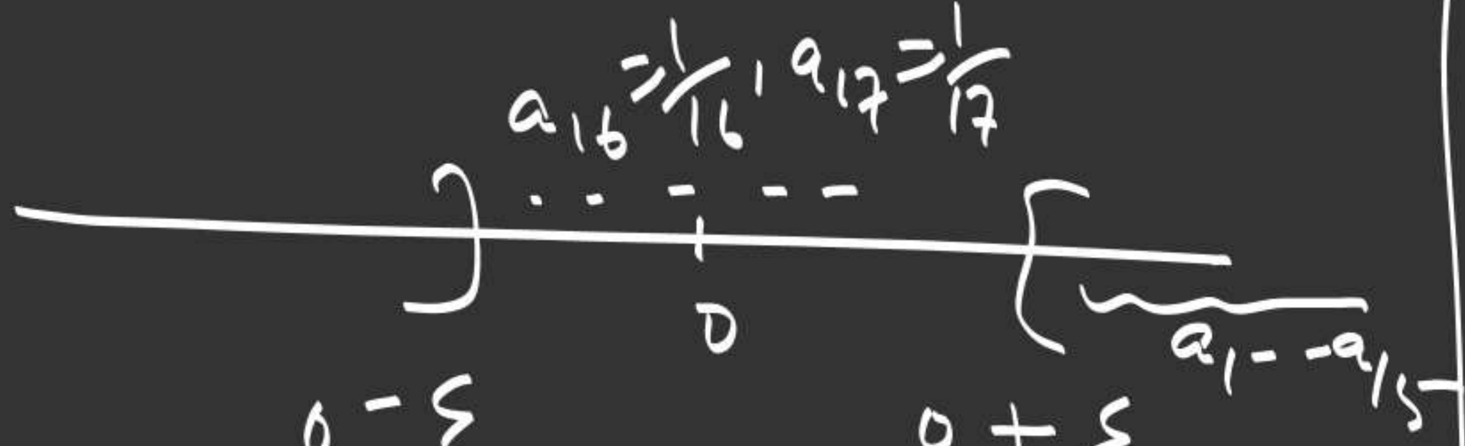
$\langle \frac{1}{n} \rangle$ $\qquad$ $a_n = \frac{1}{n}$

$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$



$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0}$$

$-\varepsilon < \frac{1}{n} < \varepsilon \; , \forall n \geq 16$



$0 - \varepsilon \quad \left(-\frac{1}{15}\right)$

$0 + \varepsilon \quad \left(\frac{1}{15}\right)$

$a_1 = 1 , \; a_2 = \frac{1}{2} - - \text{---}$

$a_{15} = \frac{1}{15}$

<u>Ex:</u> $\langle a_n \rangle , \; a_n = \frac{1}{n}$

<u>To prove</u> $\lim a_n = 0$

Choose $\varepsilon = \frac{1}{15}$

$|a_n - l| = \left|\frac{1}{n} - 0\right|$

$= \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$

if $n > \frac{1}{\varepsilon}$

i.e. $n > 15$

$\underline{\left|\frac{1}{n} - 0\right| < \varepsilon} , \; \forall n \geq 16$

(m)

Convergence of a sequence

A sequence $\langle a_n \rangle$ is said to converge to a real number l (or to have l as a limit) if

for each $\varepsilon > 0$ $\exists$ a +ve integer m (depending on ε) s.t.

$$|a_n - l| < \varepsilon \quad \forall\, n \geq m$$

Notation: $\lim\limits_{n\to\infty} a_n = l$, $\lim a_n = l$, $\langle a_n \rangle \to l$

$a_n \to l$.

Note ① For any $\varepsilon > 0$, at the most a finite number of terms (depending on the choice of ε) of the sequence can lie outside the open interval $]l-\varepsilon, l+\varepsilon[$.

Let m be a +ve integer greater than $\frac{1}{\varepsilon}$.

$\therefore \left|\frac{1}{n} - 0\right| < \varepsilon, \ \forall n \geq m.$

i.e. $\lim \frac{1}{n} = 0$.

(11) $a_n \in \,]l-\varepsilon, l+\varepsilon[$ for infinitely many values of n.

Prob: Prove that $\lim \frac{1}{n} = 0$.

Solⁿ: Let $\varepsilon > 0$ be any number (Let $\varepsilon > 0$ be given)

$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

Prob Prove that $\lim a_n = 1$,

where $a_n = 1 + \frac{(-1)^n}{n}$.

Sol^n: Let $\varepsilon > 0$ be any number

$$|a_n - 1|$$

$$= \left|\cancel{1} + \frac{(-1)^n}{n} - \cancel{1}\right| = \left|\frac{(-1)^n}{n}\right|$$

$$= \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}.$$

Let m be a +ve integer greater than $\frac{1}{\varepsilon}$.

$\therefore$ For $\varepsilon > 0$, $\exists$ a +ve integer m s.t.

$$|a_n - 1| < \varepsilon, \ \forall n \geq m.$$

$\therefore \lim a_n = 1$.

Prob: If $a_n = \sqrt{n+1} - \sqrt{n}$ then $\lim a_n = 0$.

Sol^n: Let $\varepsilon > 0$ be any number.

$$|a_n - 0| = \left|\sqrt{n+1} - \sqrt{n}\right|$$

$$= \frac{\left(\sqrt{n+1} - \sqrt{n}\right) \times \left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}} < \varepsilon$$

i.e. if $2\sqrt{n} > \frac{1}{\varepsilon}$

i.e. if $\sqrt{n} > \frac{1}{2\varepsilon}$

i.e. if $n > \frac{1}{4\varepsilon^2}$

Prob: By definition, show that $\lim_{n \to \infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$.

Solⁿ: Let $\varepsilon > 0$ be any number

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right|$$

$$= \left| \frac{3+\cancel{2\sqrt{n}}-\cancel{2\sqrt{n}}}{\sqrt{n}} \right|$$

$$= \left| \frac{3}{\sqrt{n}} \right| = \frac{3}{\sqrt{n}}$$

Let m be a +ve int. greater than $\frac{1}{4\varepsilon^2}$.

$\therefore$ for $\varepsilon > 0$, $\exists$ a +ve int. m s.t

$$|a_n - 0| < \varepsilon, \ \forall n \geq m.$$

$\therefore \lim a_n = 0$.

$\therefore \lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$

<u>Prob:</u> prove that

$$\lim \frac{1+2+3+\cdots+n}{n^2} = \frac{1}{2}.$$

<u>Sol^n:</u> Let $\varepsilon > 0$ be any number

$$\left| \frac{1+2+3+\cdots+n}{n^2} - \frac{1}{2} \right|$$

$< \varepsilon$ if $\sqrt{n} > \frac{3}{\varepsilon}$

i.e. $n > \frac{9}{\varepsilon^2}$.

Let m be a +ve integer greater than $\frac{9}{\varepsilon^2}$.

$\therefore$ For $\varepsilon > 0$, $\exists$ a +ve int. m s.t.

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon, \ \forall n \geq m$$

$\therefore$ For $\varepsilon > 0$, $\exists$ a +ve int m

s.t. $\left|\dfrac{1+2+3+\cdots+n}{n^2} - \dfrac{1}{2}\right| < \varepsilon$

$\forall n \geq m$

$\therefore \lim \dfrac{1+2+3+\cdots+n}{n^2} = \dfrac{1}{2}$.

$$= \left|\frac{\frac{n(n+1)}{2}}{n^2} - \frac{1}{2}\right|$$

$$= \left|\frac{\cancel{n}(n+1)}{2\,n^{\cancel{2}}} - \frac{1}{2}\right|$$

$$= \left|\frac{n+1-n}{2n}\right| = \frac{1}{2n} < \varepsilon$$

if

$2n > \frac{1}{\varepsilon}$

i.e. $n > \frac{1}{2\varepsilon}$.

let m be a +ve int. greater than $\frac{1}{2\varepsilon}$.

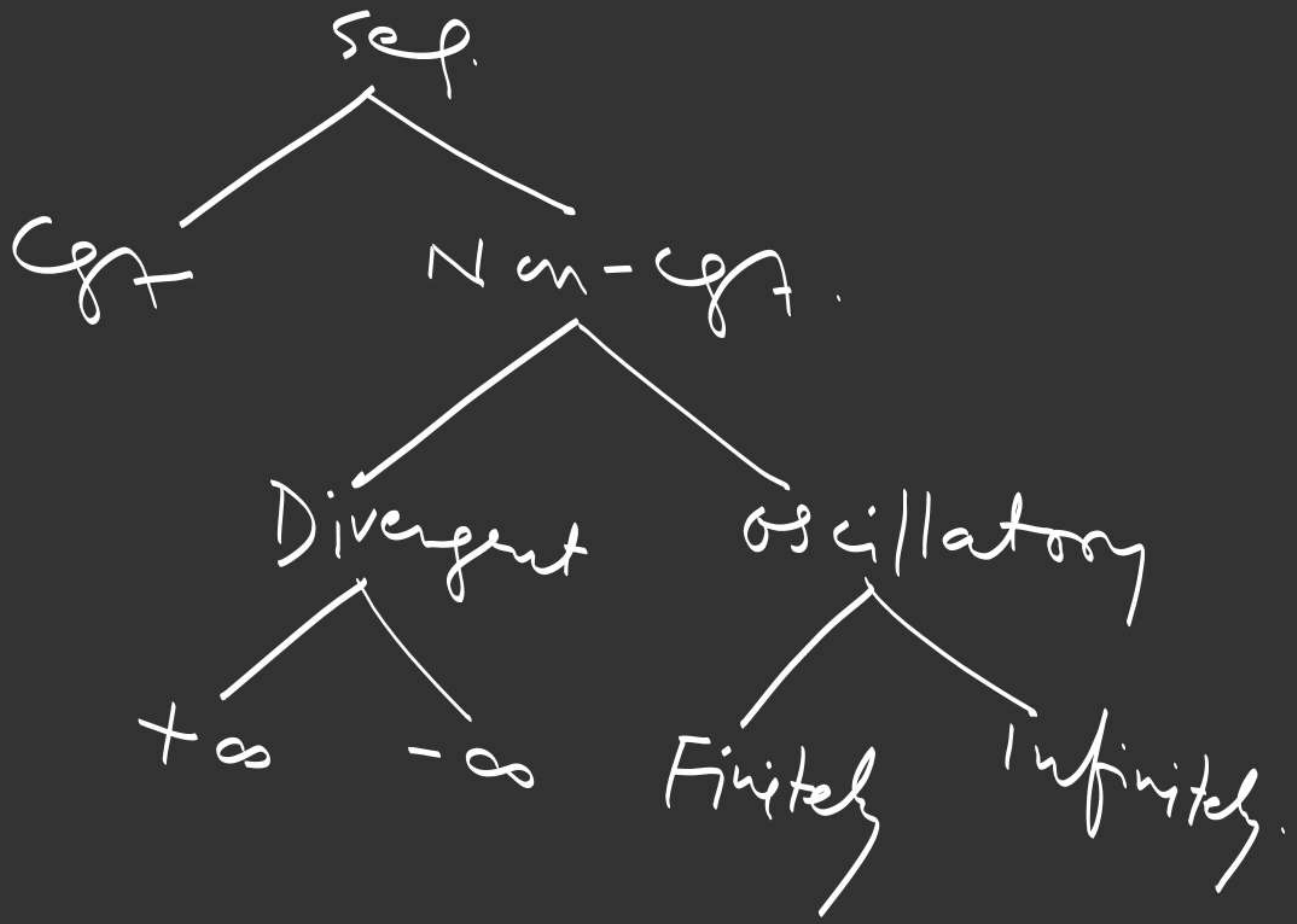
Divergent sequence

→ A sequence $\langle a_n \rangle$ is said to diverge to $+\infty$ if for a +ve number K (may be very large) $\exists$ a +ve integer m (depending on K) s.t.

$$a_n > K \quad \forall n \geq m.$$

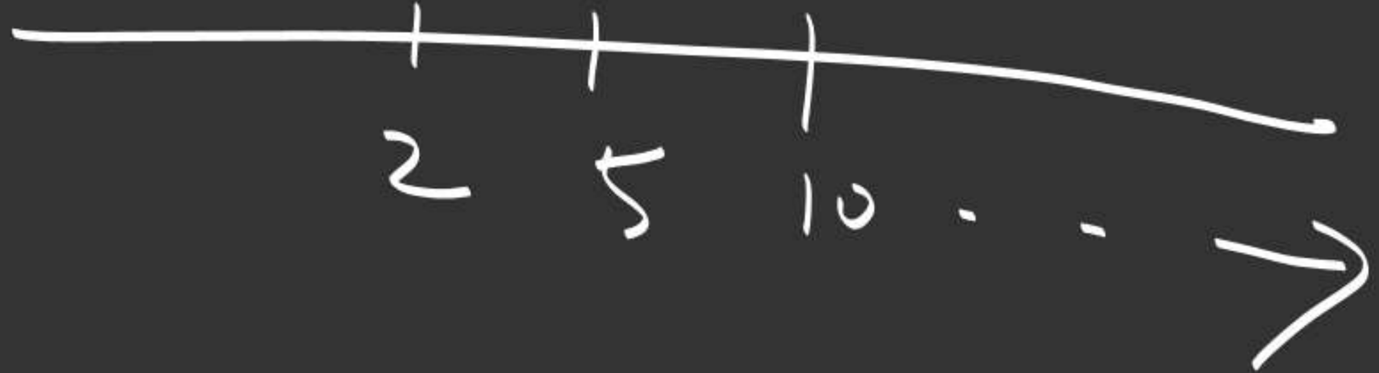
$$\lim a_n = \infty$$

Non-convergent sequence



Ex (I) $\langle n^2+1 \rangle$

$= \langle 1^2+1,\ 2^2+1,\ 3^2+1, - - \rangle$



Divergen to $+\infty$.

(II) $\langle -n^2 \rangle$

$= \langle -1^2, -2^2, -3^2, - - \rangle$

→ A seq. $\langle a_n \rangle$ is said to diverge to $-\infty$ iff for a +ve real number K (may be very large) $\exists$ a +ve int. m (Depending on K) s.t.

$$a_n < -K \quad \forall\, n \geq m$$

$$\lim a_n = -\infty$$

sequence is called oscillatory.

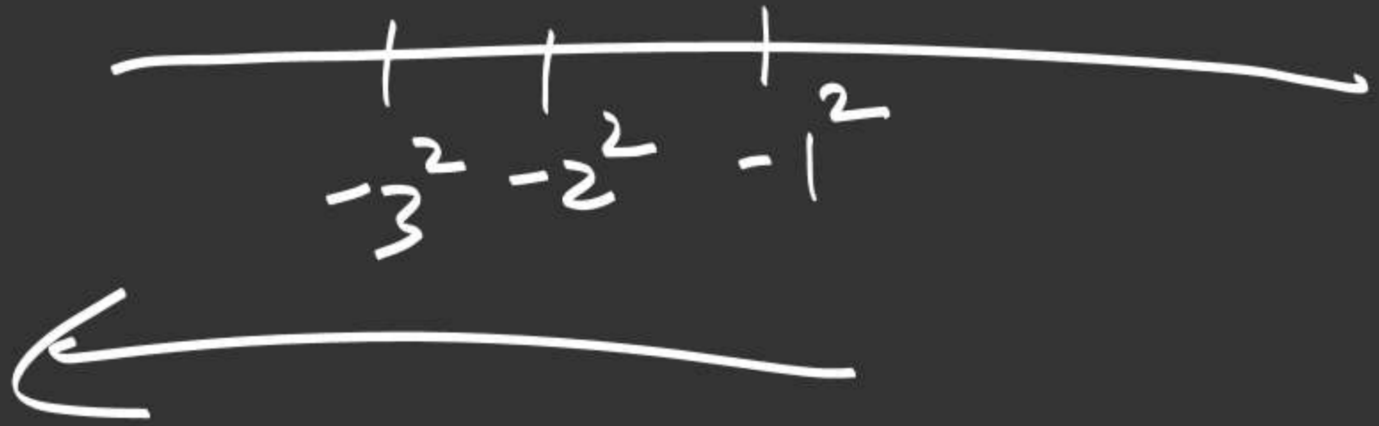
→ A sequence is said to oscillate finitely if

(i) it is oscillatory.

(ii) it is bdd.

Ex: (1) $\langle (-1)^n \rangle$

$= \langle -1, 1, -1, 1, \ldots \rangle$



Divergent to $-\infty$.

Oscillatory sequence

If a sequence is neither convergent nor divergent then the

→ A sequence is said to oscillate infinitely if

(I) it is oscillatory.

(II) it is unbounded.

Ex: $\langle (-1)^n \cdot n \rangle$

$\equiv \langle -1, 2, -3, 4, -5, \dots \rangle$

oscillate finitely.

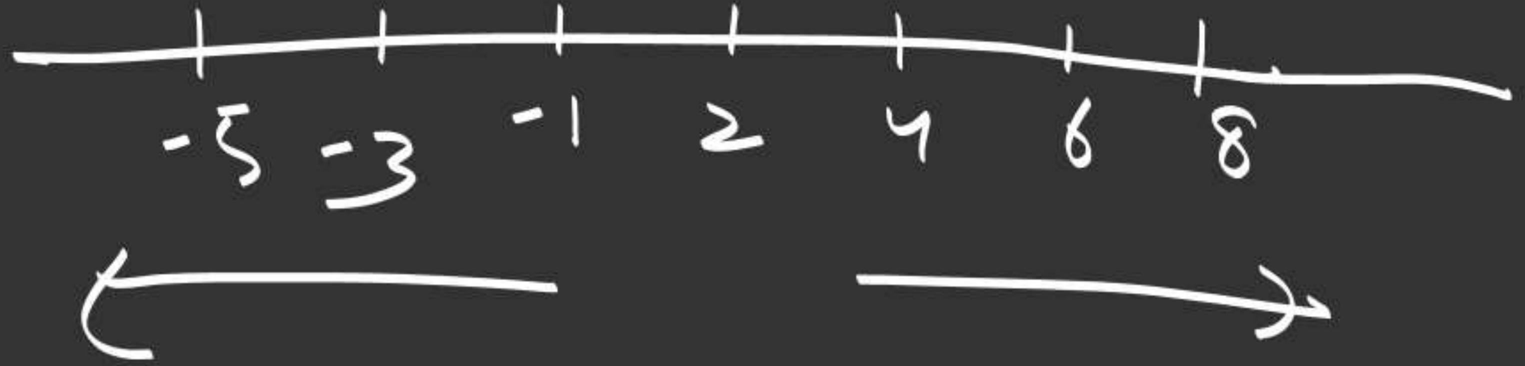
Ex: $\langle 1+(-1)^n \rangle$

$\equiv \langle 0, 2, 0, 2, 0, 2, \dots \rangle$

oscillate finitely.

<u>Ex</u> $\langle (-1)^n \cdot n^2 \rangle$

oscillate infinitely.



oscillate infinitely.

<u>Ex:</u> $\langle (-2)^n \rangle$

$= \langle -2, 4, -8, 16, \ldots \rangle$

oscillate infinitely.