KGS IAS

Maths Optional

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$$0 + f(0) = f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

put $y = -x$ in (1)

$$f(x-x) = f(x) + f(-x)$$

$$f(0) = f(x) + f(-x)$$

$$\Rightarrow 0 = f(x) + f(-x)$$

$$\Rightarrow f(-x) = -f(x) \quad \text{—— (ii)}$$

prob: If f is a continuous function of x satisfying the functional eqn.

$$f(x+y) = f(x) + f(y) \quad \text{— (1)}$$

Show that $f(x) = ax \;\; \forall x \in \mathbb{R}$ where a is a constant.

→ put $x = y = 0$ in (1)

$$f(0+0) = f(0) + f(0)$$

$$\Rightarrow f(0) = f(0) + f(0)$$

Let x is a -ve int.

Let $x = -y$, $\underline{y > 0}$

$f(x) = f(-y)$

$= -f(y)$ using (1)

$= -ay$ [$\because y$ is a +ve int.]

$= a \cdot (-y)$

$= ax$

Let x is a positive int.

$f(x) = f(1+1+1+\cdot - x \text{ times})$

$= \underbrace{f(1) + f(1) + \cdots + f(1)}_{x \text{ times}}$ [using (1)]

$= x f(1)$

$\therefore f(x) = ax$, $\boxed{a = f(1)}$

$\therefore f(x) = ax$

Let x is a real no.

$\therefore$ we can have a seq. of rational no $\langle x_n \rangle \to x$

$\because f$ is cts.

$\therefore f$ is cts at 'x'.

$\therefore \langle f(x_n) \rangle \to f(x)$

Now $\forall n \in \mathbb{N}$

$f(x_n) = a x_n \quad \forall a$ (given)

x is a rational no.

let $x = \frac{p}{q}$, $\quad q > 0$

$$f(p) = f\left(q \cdot \frac{p}{q}\right)$$

$$ap = f\left(\frac{p}{q} + \frac{p}{q} + \cdots q \text{ times}\right)$$

$$\Rightarrow ap = \underbrace{f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \cdots - f\left(\frac{p}{q}\right)}_{q \text{ times}}$$

$$\Rightarrow ap = q \cdot f\left(\frac{p}{q}\right) \quad \text{(using (1))}$$

$$\Rightarrow f\left(\frac{p}{q}\right) = a \cdot \frac{p}{q}$$

Let, if possible, f is not bdd above.

$\therefore$ For each +ve int. n

$\exists$ some $x_n \in [a, b]$ s.t.

$$f(x_n) > n$$

$\therefore$ we have a seq. $\langle x_n \rangle$ s.t

$$a \le x_n \le b \quad \forall n \in \mathbb{N}.$$

$\therefore$ $\langle x_n \rangle$ is bdd

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (a \cdot x_n)$$

$$\Rightarrow f(x) = a \cdot \lim x_n$$
$$= ax$$

$$\therefore \underline{f(x) = ax}$$

<u>Th</u>: If f is cts on $[a, b]$ then it is bdd in $[a, b]$.

<u>Proof</u>: suppose f is cts in $[a, b]$

which will converge to p.

$\because p \in [a,b]$

$\therefore f$ is cts at 'p'.

seq. $\langle f(x_{n_k}) \rangle$ where

$$f(x_{n_k}) > n_k$$

$\therefore \langle f(x_{n_k}) \rangle \to +\infty$.

which is a contradiction to the fact that f is cts at 'p'.

$\therefore \langle x_n \rangle$ has a limit point 'p' (say).

$\because [a,b]$ is a closed set.

$\therefore$ it contains all its limit points.

$\therefore p \in [a,b]$

$\because p$ is a limit pt of $\langle a_n \rangle$

$\therefore \exists$ a subsequence $\langle a_{n_k} \rangle$

Note ① The converse of the above th. is not true.

Consider the example

Define f on $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$ by

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$-1 \leq f(x) \leq 1 \quad \forall x \in \left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$$

$\therefore$ our supposition is wrong.

$\therefore$ f is bdd above.

Similarly by taking $-f$ we can show that f is bdd below.

Consider the example.

Define $f(x) = 1/x$ $\forall x \in]0,1[$

so $f(x)$ is cts on $]0,1[$.

$f(x)$ is not bdd in $]0,1[$.

(As $f(x)$ is not bdd above)

But $f(x)$ is not cts at $x = 0$

As $\lim_{x \to 0} f(x)$ doesn't exist.

(1) If f is cts on open interval then f need not be bdd.

Proof: Suppose f is a constant function.

Then obviously, f attains its bounds in [a, b].

Let f be a function which is not constant.

$\because$ f is cts on [a, b]

$\therefore$ it is bdd in [a, b]

Th: If f is cts on [a, b] then f attains its bounds.

or

If f is cts on [a, b] then f attains its supremum as well as its infimum at least once in [a, b].

$\therefore M - f(x) > 0$

Let $g(x) = \dfrac{1}{M - f(x)}$ $\forall x \in [a,b]$

$\because f(x)$ is cts on $[a,b]$

$\therefore M - f(x) \; (\neq 0)$ is also cts on $[a,b]$

$\therefore \dfrac{1}{M - f(x)}$ ''

$\therefore g(x)$ ''

$\therefore g(x)$ is bdd on $[a,b]$

Let $\inf f = m$ and $\sup f = M$

we have to find some $\alpha, \beta \in [a,b]$ s.t. $f(\alpha) = m$, $f(\beta) = M$

Let us consider the case of supremum:

Let, if possible, $f(x) \neq M$ for any $x \in [a,b]$.

$\therefore f(x) \le M - \frac{1}{K} \quad \forall x \in [a,b]$

$(K > 0)$

$\therefore M - 1/K$ is an upper bound of $f(x)$.

Also $M - \frac{1}{K} < M$

So this is a contradiction.

$\therefore$ our supposition is wrong.

$\therefore \exists$ some $\beta \in [a,b]$ s.t.

$f(\beta) = M$.

$\therefore g(x)$ is bdd above also.

$\therefore \sup g(x)$ exists

Let $\sup g(x) = K$

$\therefore g(x) \le K \quad \forall x \in [a,b]$

$$\Rightarrow \frac{1}{M - f(x)} \le K$$

$$\Rightarrow M - f(x) \ge \frac{1}{K}$$

$$\Rightarrow M - \frac{1}{K} \ge f(x)$$

$\sup f = 1$.

$\inf f = 0$

$f(0) = 0$, But $0 \notin]0,1]$.

Similarly, we can show that f attains its infimum in $[a,b]$.

<u>Note</u> ① The above th. need not be true if the interval is not closed.

$$f(x) = x \quad \forall x \in]0,1]$$

f is cts in $]0,1]$.

$f(-\frac{2}{\pi}) = -1$

$f(\frac{2}{\pi}) = 1$

But f is not cts at $x=0$

<u>Th:</u> If f is cts on $[a,b]$ then f is bdd and attains its bounds at least once in $[a,b]$.

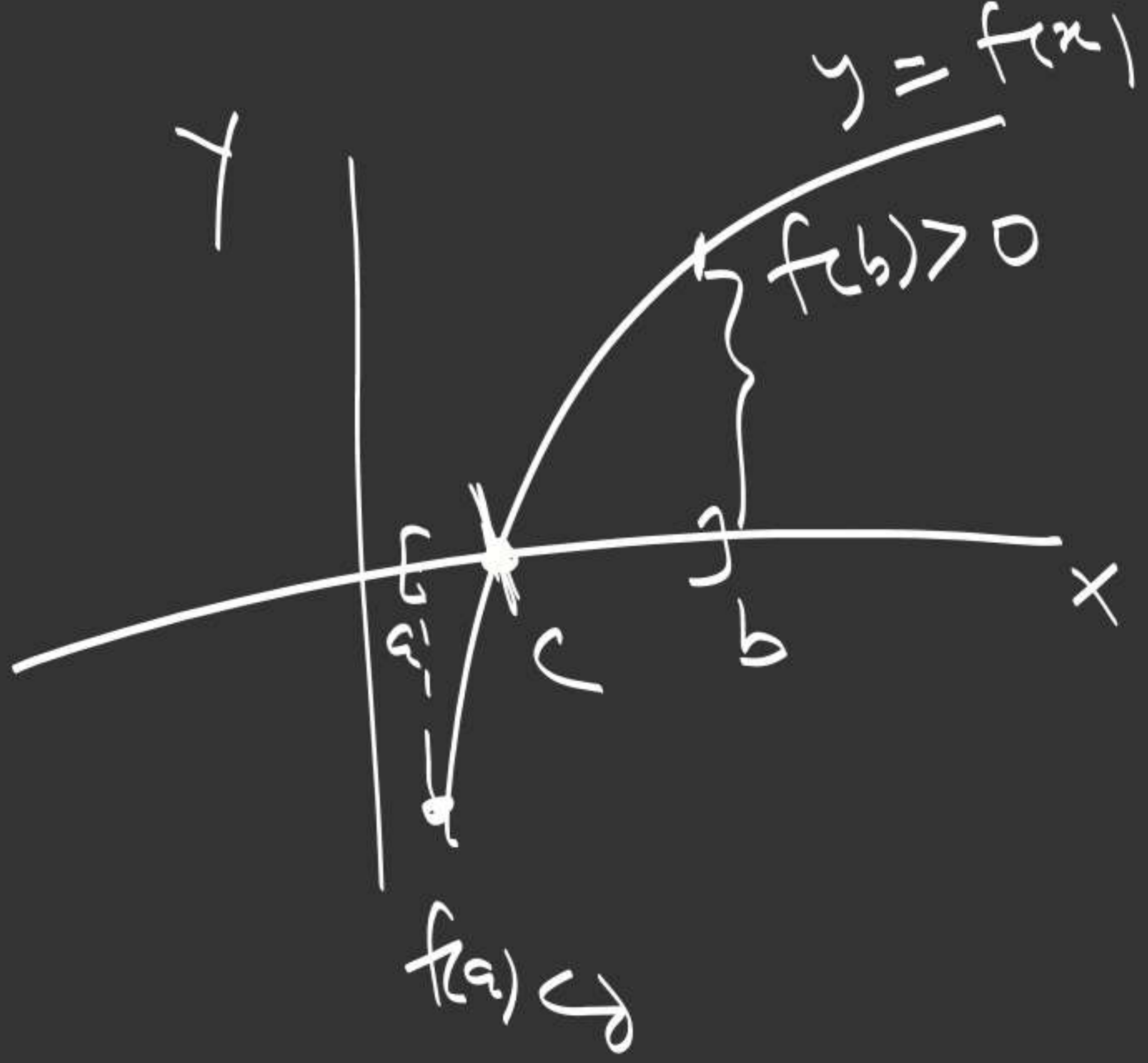
(11) Converse of the above th is not true.

Consider the following function defined on $[-\frac{2}{\pi}, \frac{2}{\pi}]$

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$-1 \leq f(x) \leq 1 \quad \forall x \in [-\frac{2}{\pi}, \frac{2}{\pi}]$

$c \in]a, b[$ s.t. $f(c) = 0$



Sign preservation th:

If f is cts on $[a, b]$ and $c \in]a, b[$ s.t. $f(c) \neq 0$ then $\exists$ some $\delta > 0$ s.t. $f(x)$ have same sign as that of $f(c)$, $\forall x \in]c-\delta, c+\delta[$.

Result: If f is cts on $[a, b]$ and $f(a)$ & $f(b)$ are of opposite signs then $\exists$ at least one pt.

Consider the example

$$f(x) = \begin{cases} x+1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

f is not cts at $x=0$

$\therefore f$ is not cts on $[0,1]$.

$f(0) = 0 \qquad f(1) = 2$

$$f(0) < 1 < f(1)$$

There is no point $c \in [0,1]$ s.t. $f(c) = 1$.

Intermediate value th. (IVT).

If f is cts on $[a,b]$ and $f(a) \neq f(b)$ then f assumes every value b/w $f(a)$ & $f(b)$.

Remark: If a function f is not cts on a closed int. then the conclusion of IVT may not hold.

Prob: Let f be cts on $[0,1]$ and let $f(x)$ be in $[0,1]$ $\forall x \in [0,1]$. Prove that $f(x) = x$ for some $x \in [0,1]$.

$\rightarrow \quad 0 \le f(x) \le 1 \quad \forall x \in [0,1]$

If $f(0) = 0$ and $f(1) = 1$, we are done.

If not, $f(0) > 0$, $f(1) < 1$

Result: If a function f is cts on $[a,b]$ then it assumes every value b/w its bounds.

Result: The image of a closed interval under a continuous fn. is closed.

$\therefore \exists$ some $c \in]0,1[$

s.t. $g(c) = 0$

$\Rightarrow f(c) - c = 0$

$\Rightarrow \underline{f(c) = c}$

Define $g(x) = f(x) - x$

on $[0,1]$

$\because f(x)$ is cts on $[0,1]$

$\therefore$ $g(x)$ is also cts on $[0,1]$.

$g(0) = f(0) - 0$

$= f(0) > 0$

$\therefore g(0) > 0$

$g(1) = f(1) - 1 < 0$

$\therefore g(1) < 0$

Let $\inf f = m$ & $\sup f = M$

$\therefore m \le f(x) \le M \quad \forall x \in [a,b]$

$\therefore m \le f(x_i) \le M \quad \forall x_i \in [a,b]$

$i = 1, 2, \dots n$

Now

$m + m + \dots n \text{ times} \le f(x_1) + f(x_2) + \dots + f(x_n)$

$\le M + M + \dots n \text{ times}$

$\Rightarrow m \cdot n \le f(x_1) + f(x_2) + \dots + f(x_n) \le nM$

prob: Let f be cts on $[a,b]$ and $x_1, x_2, \dots x_n$ be points of $[a,b]$. Show that $\exists$ some $c \in [a,b]$ s.t.

$$f(c) = \frac{1}{n}\left[f(x_1) + f(x_2) + \dots f(x_n)\right]$$

$\rightarrow \because f$ is cts on $[a,b]$

$\therefore f$ is bdd ,,

prob: let $f(x) = \lim_{n\to\infty} \dfrac{\ln(2+x) - x^{2n}\cdot \sin x}{1+x^{2n}}$

, $x \geq 0$. Show $f(0)$ &

$f(\pi/2)$ differ in sign.

why does $f(x)$ not vanish

in $[0, \pi/2]$? Explain.

$\rightarrow$

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$$

$$m \leq \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right] \leq M$$

$\therefore \exists\ c \in [a, b]$

s.t.

$$f(c) = \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

Also

$f(x) \neq 0, \forall x \in [0, \pi/2]$

Reason: $1 \in [0, \pi/2]$

f is not cts at $x = 1$

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$$f(x) = \begin{cases} \dfrac{\log(2+x)}{}, & 0 \le x < 1 \\ \dfrac{\log 3 - \sin 1}{2}, & x = 1 \\ \underline{-\sin x}, & x > 1 \end{cases}$$

$f(0) = \log 2 > 0$

$f(\pi/2) = -\sin \pi/2$

$= -1 < 0$