Maths Optional

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prob: use sequential criterion to prove that

$$\lim_{x \to 2} \frac{1}{1-x} = -1$$

$\to$ Take $x_n = \frac{2n}{n+1}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n}{n+1}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = 2$$

Sequential criterion

Let $f : A \to \mathbb{R}$ and let 'c' be a limit point of A then the followings are equivalent

(I) $\lim_{x\to c} f(x) = l$

(II) For every seq. $\langle x_n \rangle$ in A ($x_n \neq c$ $\forall n$) converging to c $\langle f(x_n) \rangle \to l$.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1+n}{1-n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n}-1}$$

$$= \frac{0+1}{0-1} = -1.$$

$$\therefore \lim_{x\to 2} f(x) = -1.$$

$$f(x) = \frac{1}{1-x}$$

$$f(x_n) = \frac{1}{1-x_n}$$

$$= \frac{1}{1-\frac{2n}{n+1}}$$

$$= \frac{n+1}{n+1-2n} = \frac{n+1}{1-n}$$

iff $\exists$ a seq. $\langle x_n \rangle \subseteq A$ $(x_n \neq c)$ converging to c but $\langle f(x_n) \rangle$ diverges (does not converge)

Prob: $\lim_{x \to 0} \frac{1}{x}$ doesn't exist.

$\to$ Consider $\langle x_n \rangle$ where $x_n = \frac{1}{n}$

$\therefore \lim x_n = 0$

but $f(x) = \frac{1}{x}$

Divergence criteria

$A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, c is a limit point of A.

(1) f doesn't have a limit l at $x = c$ iff $\exists$ a seq. $\langle x_n \rangle$ in A $(x_n \neq c)$ converging to c but $\langle f(x_n) \rangle \not\to l$.

(1) f doesn't have a limit at $x = c$

<u>prob:</u> $\lim_{x\to 0} \frac{1}{x^2}$ doesn't exist.

$\to$ Consider $x_n = \frac{1}{n}$

$\therefore \lim x_n = \lim_{n\to\infty} \frac{1}{n} = 0$

let $f(x) = \frac{1}{x^2}$

$f(x_n) = n^2$

$\lim f(x_n) = \infty$.

$\langle f(x_n) \rangle$ diverges.

$f(x_n) = \frac{1}{x_n}$

i.e $f(\frac{1}{n}) = \frac{1}{\frac{1}{n}} = n$

$\lim f(\frac{1}{n}) = \infty$.

$\therefore \langle f(x_n) \rangle$ diverges.

$\therefore \lim_{x\to 0} \frac{1}{x}$ doesn't exist.

$$\lim x_n = \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

$$f(x_n) = \frac{x_n}{|x_n|}$$

$$= \frac{\frac{(-1)^n}{n}}{\left|\frac{(-1)^n}{n}\right|} = \frac{\frac{(-1)^n}{\cancel{n}}}{\frac{1}{\cancel{n}}}$$

$$f(x_n) = (-1)^n$$

$\langle f(x_n) \rangle$ is oscillatory.

$\therefore \lim_{x\to 0} \frac{1}{x^2}$ doesn't exist.

<u>prob</u>: $\lim_{x\to 0} \operatorname{sgn}(x)$ doesn't exist.

$\to$ Let $f(x) = \operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Consider $x_n = \frac{(-1)^n}{n}$.

Prob: $\lim\limits_{x \to 0} \sin \frac{1}{x}$ doesn't exist.

$\longrightarrow$ consider

$$x_n = \frac{1}{n\pi} \qquad y_n = \frac{1}{2n\pi + \pi/2}$$

$$\lim x_n = \lim_{n \to \infty} \frac{1}{n\pi} = 0$$

$$\lim y_n = \lim_{n \to \infty} \frac{1}{2n\pi + \pi/2} = 0$$

Let $f(x) = \sin \frac{1}{x}$

$$\langle f(x_n) \rangle \equiv \langle \sin n\pi \rangle \to 0$$

$$\langle f(y_n) \rangle \equiv \langle \sin(2n\pi + \pi/2) \rangle \longrightarrow 1$$

$\therefore \lim\limits_{x \to 0} \sin \frac{1}{x}$ doesn't exist.

If $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$ then

(i) $\lim_{x \to c} (f \pm g) = l \pm m$

(ii) $\lim_{x \to c} (f \cdot g) = l \cdot m$.

(iii) $\lim_{x \to c} (K f) = K \cdot l$.

(iv) $\lim_{x \to c} \left(\frac{f}{g}\right) = \frac{l}{m}$, $m \neq 0$

HW **prob:** $\lim_{x \to 0} \sin \frac{1}{x^2}$ doesn't exist.

$\rightarrow$ $x_n = \frac{1}{\sqrt{n\pi}}$ $\quad y_n = \frac{1}{\sqrt{2n\pi + \frac{\pi}{2}}}$

Algebra of limits: Let $A \subseteq \mathbb{R}$, f and g are defined on A, c is a limit point of A.

and 'c' be a limit point of A.

If $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$, $x \neq c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$

then $\lim_{x \to c} g(x) = l$ (provided $\lim_{x \to c} g(x)$ exists).

Result: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$

Let c is a limit point of A.

If $a \leq f(x) \leq b$ $\forall x \in A$, $x \neq c$

then $a \leq \lim_{x \to c} f(x) \leq b$.

(provided $\lim_{x \to c} f(x)$ exists)

Squeeze th: Let $A \subseteq \mathbb{R}$,

Let $f, g, h : A \to \mathbb{R}$

Prob: $\lim_{x \to 0} \sin x = 0$

$\to$ $-x \le \sin x \le x$

$x \geq 0$

which is of the form

$f(x) \le g(x) \le h(x)$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$

$\therefore \lim_{x \to 0} \sin x = 0.$

Note: $x \in \mathbb{R},\ x > 0$

1. $-x \le \sin x \le x$
2. $1 - \frac{1}{2}x^2 \le \cos x \le 1.$
3. $x - \frac{x^3}{6} \le \sin x \le x.$
4. $1 - \frac{1}{2}x^2 \le \cos x \le 1 - \frac{1}{2}x^2 + \frac{x^4}{24}$

$\lim_{x \to 0} h(x) = 1$

$\therefore$ By squeeze th,

$\lim_{x \to 0} g(x) = 1$

i.e. $\lim_{x \to 0} \cos x = 1$.

Prob: $\lim_{x \to 0} \cos x = 1$

$\rightarrow$ $\forall x \geq 0$

$$1 - \frac{x^2}{2} \leq \cos x \leq 1$$

which is of the form

$$f(x) \leq g(x) \leq h(x)$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(1 - \frac{x^2}{2}\right) = 1$$

$f(x) \le g(x) \le h(x)$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$

$\therefore$ By Squeeze Th.,

$\underline{\lim_{x \to 0} x^{3/2} = 0}$

Prob: Show that

$$\lim_{x \to 0} x^{3/2} = 0 \quad (x > 0)$$

$\to$ For $0 < x \le 1$

$\Rightarrow x \le x^{1/2} \le 1$

$\Rightarrow x^2 \le x^{3/2} \le x \quad$ (As $x > 0$)

which is of the form

$$f(x) \le g(x) \le h(x)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 1$$

$\therefore$ By Squeeze th.

$$\lim_{x\to 0} g(x) = 1$$

$$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$$

<u>Prob:</u> Show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$\rightarrow$ For $x > 0$

$$x - \frac{x^3}{6} \le \sin x \le x$$

$$\Rightarrow 1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad (x > 0)$$

Which is of the form

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$$

$\therefore$ By squeeze th.

$$\lim_{x\to 0} g(x) = 0$$

i.e. $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x} = 0$

prob: $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x} = 0$

$\rightarrow \quad -1 \le \sin\frac{1}{x} \le 1 \qquad \underline{x \neq 0}$

$-x \le x\sin\frac{1}{x} \le x, \qquad \underline{x > 0}$

which is of the form

$$f(x) \le g(x) \le h(x)$$

As $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist.

$\therefore \lim_{x \to 0} \operatorname{Sgn}\left(\sin\frac{1}{x}\right)$ doesn't exist.

<u>Prob</u> Show that

$\lim_{x \to 0} \operatorname{Sgn}\left(\sin\frac{1}{x}\right)$ doesn't exist.

$\longrightarrow$ $\operatorname{Sgn}\left(\sin\frac{1}{x}\right) = \dfrac{\sin\frac{1}{x}}{\left|\sin\frac{1}{x}\right|}$

$$\lim_{x \to 0} \operatorname{Sgn}\left(\sin\frac{1}{x}\right) = \lim_{x \to 0} \frac{\sin\frac{1}{x}}{\left|\sin\frac{1}{x}\right|}$$

if for any $\varepsilon > 0$, $\exists$ $\delta > 0$ (depending on ε) s.t.

$|f(x) - l| < \varepsilon$ whenever

$c < x < c + \delta$

& $x \in A$

or

f tends to approach l as $x \to c$ from right if for any $\varepsilon > 0$ $\exists$ some $\delta > 0$

one sided limit:

Right hand limit: Let $A \subseteq \mathbb{R}$

$f : A \to \mathbb{R}$ if c is a limit pt. of $A \cap]c, \infty[$ $= \{x \in A : x > c\}$ then we say that l is the right hand limit of $f(x)$ at 'c'

Note: $\lim_{x \to c+} f(x)$ exists despite f being not defined at $x = c$.

Left hand limit: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c is a limit point of $A \cap]-\infty, c[= \{x \in A : x < c\}$ then we say that l is the left hand limit of f at 'c' if

s.t.

$|f(x) - l| < \varepsilon$ whenever $c < x < c + \delta$

Notation: $\lim_{x \to c+0} f(x) = l$

or $\lim_{x \to c+} f(x) = l$ or $\lim_{\substack{x \to c \\ x > c}} f(x) = l$

or $f(c+0) = l$ or $f(c+) = l$.

Notation: $\lim_{x \to c-0} f(x) = l$ or

$\lim_{x \to c-} f(x) = l$ or $\lim_{\substack{x \to c \\ x < c}} f(x) = l$

or $f(c-0) = l$ or $f(c-) = l$.

Note: $\lim_{x \to c-} f(x)$ can exist without f being defined at $x = c$.

for any $\varepsilon > 0$, $\exists$ $\delta > 0$

s.t. $|f(x) - l| < \varepsilon$ whenever $c - \delta < x < c$, $x \in A$.

or

$f(x)$ tends to l as $x \to c$ from left if for any $\varepsilon > 0$ $\exists$ some $\delta > 0$ s.t.

$|f(x) - l| < \varepsilon$ whenever $c - \delta < x < c$

$$\lim_{x\to 0+}\frac{|x|}{x} = \lim_{\substack{x\to 0\\ x>0}}\frac{|x|}{x}$$

$$= \lim_{\substack{x\to 0\\ x>0}}\frac{x}{x} = 1$$

$$\therefore \lim_{x\to 0-}\frac{|x|}{x} \neq \lim_{x\to 0+}\frac{|x|}{x}$$

$\therefore \lim\limits_{x\to 0}\frac{|x|}{x}$ doesn't exist.

Existence of limit

$\lim\limits_{x\to c} f(x) = l$ exists iff

$$\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) = l.$$

Ex: Does $\lim\limits_{x\to 0}\frac{|x|}{x}$ exist?

$$\to \lim_{x\to 0-}\frac{|x|}{x} = \lim_{\substack{x\to 0\\ x<0}}\frac{|x|}{x} = \lim_{\substack{x\to 0\\ x<0}}\frac{-x}{x} = -1$$