Chapter 04 — Trigonometric Equation

CONTENT

- Trigonometric equation :
- Solution of trigonometric equation
- General solution
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1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

Ex. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\ldots\ldots\ldots$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :

(i) Principal solution

(ii) General solution.

Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions.

Example. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.

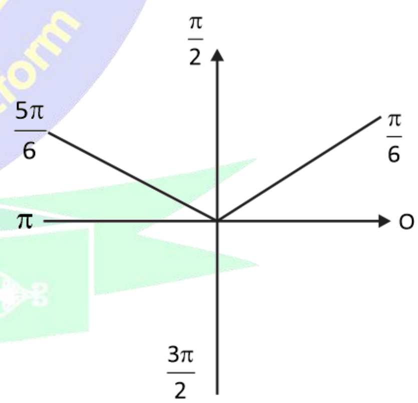
Solution :

$\because$ $\sin x = \frac{1}{2}$

$\because$ there exists two values

i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$



3. GENERAL SOLUTION :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution. General solution of some standard trigonometric equations are given below.

GENERAL SOLUTION OF SOME STANDARD TRIGONOMETRIC EQUATIONS :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

[**Note:** α is called the principal angle]

Some Important deductions :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2}, n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2}, n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2}, n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi, n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in I$

SOLVED EXAMPLES

Example: 1 Solve $\cos\theta = \frac{1}{2}$

Solution : $\because \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

Example: 2 Solve: $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Solution : $\because \sec 2\theta = -\frac{2}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow \cos 2\theta = \cos\frac{5\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I \Rightarrow \theta = n\pi \pm \frac{5\pi}{12}, n \in I$

Example: 3 Solve: $\tan\theta = \frac{3}{4}$

Solution : $\because \tan\theta = \frac{3}{4}$ ....(i)

Let $\frac{3}{4} = \tan\alpha \Rightarrow \tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha$, where $\alpha = \tan^{-1}\left(\frac{3}{4}\right), n \in I$

Example: 4 Solve $\tan^2\theta = 1$

Solution : $\because \tan^2\theta = 1 \Rightarrow \tan^2\theta = (1)^2$

$\Rightarrow \tan^2\theta = \tan^2\frac{\pi}{4} \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$

Example: 5 Solve $4\sec^2\theta = 5 + \tan^2\theta$ for θ

Solution : $\because\ 4\sec^2\theta = 5 + \tan^2\theta$ ....(i)

For equation (i) to be defined $\theta \neq (2n+1)\dfrac{\pi}{2}$, $n \in I$

$\because$ equation (i) can be written as:

$4(1 + \tan^2\theta) = 5 + \tan^2\theta$

$3\tan^2\theta = 1$

$\tan^2\theta = \tan^2\dfrac{\pi}{6}$

$\theta = n\pi \pm \dfrac{\pi}{6}$, $n \in I$

4. TYPES OF TRIGONOMETRIC EQUATIONS :

Type -1

Trigonometric equations which can be solved by use of factorization.

Example : $\dfrac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2+\sin x} = \dfrac{\cos x}{3}$

Solution : $\dfrac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2+\sin x} = \dfrac{\cos x}{3} \Rightarrow \dfrac{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2}\right)}{2+\sin x} = \dfrac{\cos x}{3}$

$\dfrac{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)(2+\sin x)}{2(2+\sin x)} = \dfrac{\cos x}{3} \Rightarrow 3\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - 2\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) = 0$

$\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\left(3 + 2\sin\frac{x}{2} + 2\cos\frac{x}{2}\right) = 0 \Rightarrow \sin\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \tan\frac{x}{2} = 1$

$\dfrac{x}{2} = n\pi + \dfrac{\pi}{4}$, $n \in I \Rightarrow x = 2n\pi + \dfrac{\pi}{2}, n \in I$

Type - 2

Trigonometric equations which can be solved by reducing them in quadratic equations.

Example : Solve $\sin^2 x - \dfrac{\cos x}{4} = \dfrac{1}{4}$

Solution : $\sin^2 x - \dfrac{\cos x}{4} = \dfrac{1}{4}$

$4(1 - \cos^2 x) - \cos x = 1$

$4\cos^2 x + \cos x - 3 = 0$

$(\cos x + 1)(4\cos x - 3) = 0$

$\cos x = -1$, $\cos x = \dfrac{3}{4}$

$x = (2n+1)\pi$, $x = (2m\pi \pm \alpha)$ where $\alpha = \cos^{-1}\left(\frac{3}{4}\right)$ $m, n \in I$

Type – 3

Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product.

Example : Solve $\cos x + \cos 3x - 2\cos 2x = 0$

Solution : $\cos x + \cos 3x - 2\cos 2x = 0$

$2\cos 2x \cos x - 2\cos 2x = 0$

$2\cos 2x (\cos x - 1) = 0$

$\cos 2x = 0, \cos x = 1$

$x = (2n + 1)\frac{\pi}{2}, x = 2m\pi$ $m, n \in I$

Type - 4

Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference.

Example : Solve $\sec 4\theta - \sec 2\theta = 2$

Solution : $\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2$

$\cos 2\theta - \cos 4\theta = 2\cos 4\theta \cos 2\theta$

$\cos 2\theta - \cos 4\theta = \cos 6\theta + \cos 2\theta$

$\cos 6\theta + \cos 4\theta = 0$

$2\cos 5\theta \cos\theta = 0$

$\cos 5\theta = 0$ or $\cos\theta = 0$

$5\theta = (2n + 1)\frac{\pi}{2}; \theta = (2m + 1)\frac{\pi}{2}$ $m, n \in I$

Type - 5

Trigonometric Equations of the form $a \sin x + b \cos x = c$, where $a, b, c \in R$, can be solved by dividing both sides of the equation by $\sqrt{a^2 + b^2}$.

Example Solve $\sin x + 2\cos x = \sqrt{5}$

Solution : $\because$ $\sin x + 2\cos x = \sqrt{5}$ ....(i)

Here a = 1, b = 2.

$\therefore$ divide both sides of equation (i) by $\sqrt{5}$, we get

$\sin x . \frac{1}{\sqrt{5}} + 2\cos x . \frac{1}{\sqrt{5}} = 1 \Rightarrow$ $\sin x.\sin\alpha + \cos x.\cos\alpha = 1$ $\Rightarrow \cos(x - \alpha) = 1$

$\Rightarrow$ $x - \alpha = 2n\pi, n \in I \Rightarrow$ $x = 2n\pi + \alpha, n \in I$

$\therefore$ Solution of given equation is $2n\pi + \alpha, n \in I$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$

DETECTIVE MIND

Trigonometric equation of the form $a \sin x + b \cos x = c$ can also be solved by changing sinx and cosx into their corresponding tangent of half the angle.

Example : Solve $3\cos x + 4\sin x = 5$

Solution : $\because$ $3\cos x + 4\sin x = 5$ ....(i)

$\because$ $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$\Rightarrow$ $3\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 4\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 5$ ....(ii)

Let $\tan\frac{x}{2} = t$

$\therefore$ Equation (ii) becomes $3\left(\dfrac{1-t^2}{1+t^2}\right) + 4\left(\dfrac{2t}{1+t^2}\right) = 5$

$\Rightarrow 4t^2 - 4t + 1 = 0$ $\Rightarrow (2t-1)^2 = 0$

$\Rightarrow t = \frac{1}{2}$ $\because$ $t = \tan\frac{x}{2}$

$\Rightarrow \tan\frac{x}{2} = \frac{1}{2} \Rightarrow$ $\tan\frac{x}{2} = \tan\alpha$, where $\tan\alpha = \frac{1}{2}$

$\Rightarrow \frac{x}{2} = n\pi + \alpha \Rightarrow x = 2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$, $n \in I$

Type - 6

Trigonometric equations of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$, where $p(y, z)$ is a polynomial, can be solved by using the substitution $\sin x \pm \cos x = t$.

Example : Solve $\sin 2x + 3\sin x = 1 + 3\cos x$

Solution : $\sin 2x + 3\sin x = 1 + 3\cos x$

$\sin 2x + 3(\sin x - \cos x) = 1$ .... (i)

Let $\sin x - \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x.\cos x = t^2 \Rightarrow \sin 2x = 1 - t^2$

Now put $\sin x - \cos x = t$ and $\sin 2x = 1 - t^2$ in (i)

$1 - t^2 + 3t = 1$

$t^2 - 3t = 0$

$t = 0$ or $t = 3$ (not possible)

$\sin x - \cos x = 0$

$\tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$, $n \in I$

Type - 7

Trigonometric equations which can be solved by the use of boundness of the trigonometric ratios sinx and cosx.

Example : Solve $\sin 2x + \cos 4x = 2$

Solution : $\sin 2x + \cos 4x = 2$

Now equation will be true if $\sin 2x = 1$ and $\cos 4x = 1$

$\Rightarrow 2x = (4n + 1)\frac{\pi}{2}, n \in I$ and $4x = 2m\pi, m \in I$

$\Rightarrow x = (4n + 1)\frac{\pi}{4}, n \in I$ and $x = \frac{m\pi}{2}, m \in I \Rightarrow (4n + 1)\frac{\pi}{4} = \frac{m\pi}{2} \Rightarrow m = \frac{4n+1}{2}$

Which is not possible for $m, n \in I$

IMPORTANT POINTS :

1. Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of $n = ...-2, -1, 0, 1, 2, 3.......$ etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

2. While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation $\tan x = 2 \sin x$. Here by dividing both sides by sinx, we get $\cos x = \frac{1}{2}$. This is not equivalent to the original equation. Here the roots obtained by $\sin x = 0$, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

3. While equating one of the factors to zero, take care of the other factor that it should not become infinite. For example, if we have the equation $\sin x = 0$, which can be written as $\cos x \tan x = 0$. Here we cannot put $\cos x = 0$, since for $\cos x = 0$, $\tan x = \frac{\sin x}{\cos x}$ is infinite.

4. **Avoid squaring** : When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.

For example : Consider the equation,

$\sin\theta + \cos\theta = 1$ .....(1)

Squaring we get

$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .....(2)

i.e. $2\theta = n\pi$ or $\theta = \frac{n\pi}{2}$,

This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$

Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$

and sin $\frac{3\pi}{2}$ + cos $\frac{3\pi}{2}$ = – 1, $\neq$ 1.

The reason for this is simple.

The equation (2) is not equivalent to (1) and (2) contains two equations : sin θ + cos θ = 1 and sinθ + cos θ = – 1. Therefore we get extra solutions.

Thus if squaring is must, verify each of the solution.

5. Some necessary restrictions :

If the equation involves tanx, secx, take cosx $\neq$ 0. If cot x or cosec x appear, take sinx $\neq$ 0.

If log appear in the equation, i.e. log [f(θ)] appear in the equation, use f(θ) > 0 and base of log > 0, $\neq$ 1.

Also note that $\sqrt{[f(\theta)]}$ is always positive, for example $\sqrt{\sin^2\theta}$ = |sin θ|, not ± sin θ.

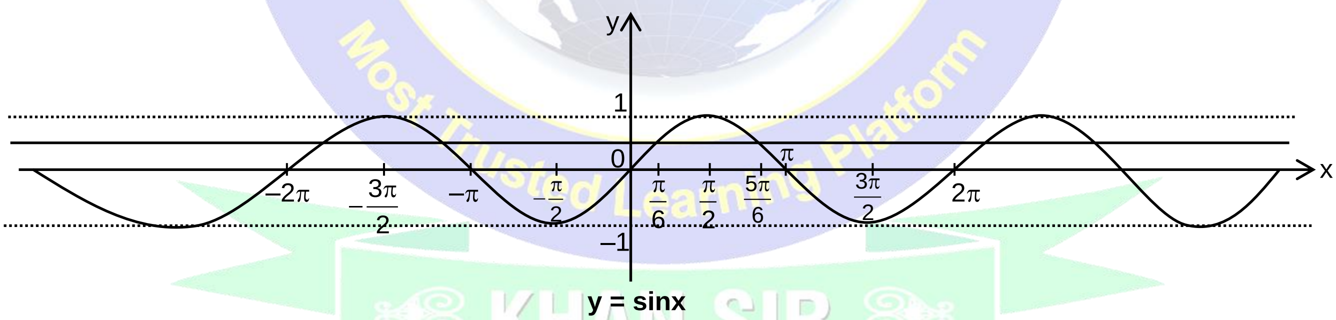
6. **Verification** : Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

5. TRIGONOMETRIC INEQUALITIES :

To solve a trigonometric inequality, transform it into many basic trigonometric inequalities. The transformation process proceeds exactly the same as in solving trigonometric equations. The common period of a trigonometric inequality is the least common multiple of all periods of the trigonometric functions presented in the inequality. **For example :** the trigonometric inequality sinx + sin2x + cos $\frac{x}{2}$ < 1 has 4π as common period. Unless specified, the solution set of a trigonometric inequality must be solved, at least within one whole common period.

Example : Find the solution set of inequality sinx > $\frac{1}{2}$.

Solution : When sinx = $\frac{1}{2}$, the two values of x between 0 and 2π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



From the graph of y = sinx, it is obvious that, between 0 and 2π, sinx > $\frac{1}{2}$ $\Rightarrow$ $\frac{\pi}{6}$ < x < $\frac{5\pi}{6}$.

Hence, sinx > $\frac{1}{2}$ $\Rightarrow$ 2nπ + $\frac{\pi}{6}$ < x < 2nπ + $\frac{5\pi}{6}$, n $\in$ I .

The required solution set is $\bigcup_{n\in I}$ (2nπ + $\frac{\pi}{6}$, 2nπ + $\frac{5\pi}{6}$)

SOLVED EXAMPLES

Example: 6 If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is

(1) $2n\pi \pm \frac{\pi}{3}$ (2) $2n\pi \pm \frac{\pi}{6}$ (3) $n\pi \pm \frac{\pi}{3}$ (4) None of these

Solution : Since $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \cos^2\theta = \frac{1}{6}\sin\theta \,.\, \tan\theta$ $\Rightarrow$ $6\cos^3\theta + \cos^2\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ satisfies the equation (by trial),

$\therefore$ $(2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \cos\theta = \cos\frac{\pi}{3}$ $\Rightarrow$ $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$.

Example :7 If $\tan\theta + \tan 2\theta + \tan\theta.\tan 2\theta = 1$, then the general value of θ is

(1) $\frac{n\pi}{3}+\frac{\pi}{12}$ (2) $\frac{n\pi}{3}-\frac{\pi}{12}$ (3) $\frac{2n\pi}{3}+\frac{\pi}{12}$ (4) $\frac{2n\pi}{3}-\frac{\pi}{12}$

Solution : we can write the equation as; $\tan\theta + \tan 2\theta = 1 - \tan\theta \,.\, \tan 2\theta$

$\Rightarrow \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta} = 1$ $\Rightarrow \tan 3\theta = 1 = \tan\frac{\pi}{4}$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{4}$ $\Rightarrow \theta = \frac{n\pi}{3}+\frac{\pi}{12}$

Example: 8 The number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ are

(1) 1 (2) 2 (3) 3 (4) 4

Solution : Here,

$\tan x + \sec x = 2\cos x$ $\Rightarrow \sin x + 1 = 2\cos^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow \sin x = \frac{1}{2}, -1$

But $\sin x = -1$ $\Rightarrow$ $x = \frac{3\pi}{2}$ which does not satisfy $\tan x + \sec x = 2\cos x$.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ Number of solutions of $\tan x + \sec x = 2\cos x$ is 2.

Example: 9 The general set of $4\sin^2 x - 8\sin x + 3 \le 0$ in the interval $[0, 2\pi]$ are

(1) $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$ (2) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (3) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ (4) None of these

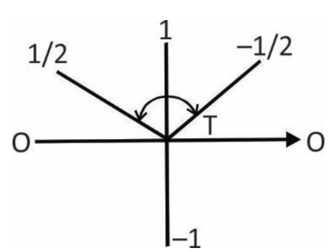
Solution : Here,

$4\sin^2 x - 8\sin x + 3 \le 0$ $\Rightarrow (2\sin x - 1)(2\sin x - 3) \le 0$

Here $2\sin x - 3$ is always negative.

$\therefore$ $2\sin x - 1 \ge 0$ i.e., $\sin x \ge \frac{1}{2}$

$\therefore$ from the figure $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$



Example: 10 The general solution of the equations $\sin^4 x + \cos^4 x = \sin x \cos x$ is -

(1) $\frac{(4n+1)\pi}{4}$ (2) $\frac{(4n-1)\pi}{4}$ (3) $\frac{(2n-1)\pi}{2}$ (4) $\frac{(2n+1)\pi}{2}$

Solution : The given equation can be written as

$4\sin^4 x + 4\cos^4 x = 4 \sin x \cos x$

or, $(1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 2\sin 2x$

or, $2(1 + \cos^2 2x) = 2\sin 2x$ or $1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow \sin^2 2x + \sin 2x = 2$

This relation is possible only if $\sin 2x = 1$

or, $2x = 2n\pi + \frac{\pi}{2} \Rightarrow x = n\pi + \frac{\pi}{4} = \frac{(4n+1)\pi}{4}$ $(n \in I)$

Example: 11 If $3 \sin^2 x + 10 \cos x - 6 = 0$ the general value of x is

(1) $n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (2) $2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

(3) $2n\pi \pm \cos^{-1}(3)$ (4) $n\pi \pm \cos^{-1}(3)$

Solution : The given equation is equivalent to $3(1 - \cos^2 x) + 10 \cos x - 6 = 0$

$\Rightarrow 3 \cos^2 x - 10 \cos x + 3 = 0 \Rightarrow (3 \cos x - 1)(\cos x - 3) = 0$

Therefore, $\cos x = \frac{1}{3}$ (because $\cos x \neq 3$). Hence $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)\frac{\pi}{2}$.

Example: 12 If $\tan x - \sin x = 1 - \tan x . \sin x$, then the general value of x is

(1) $2n\pi + \frac{\pi}{2}$ (2) $2n\pi - \frac{\pi}{2}$ (3) $n\pi + \frac{\pi}{4}$ (4) $n\pi - \frac{\pi}{4}$

Solution : Here, $\tan x - \sin x = 1 - \tan x . \sin x$

$\Rightarrow \tan x + \tan x . \sin x - \sin x - 1 = 0$

$\Rightarrow \tan x (1 + \sin x) - (1 + \sin x) = 0$

$\Rightarrow (\tan x - 1)(1 + \sin x) = 0$

$\tan x = 1$ or $\sin x = -1$

$\Rightarrow x = n\pi + \frac{\pi}{4}$ or $x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$

But for the second solution $x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$

tan x is not defined. This solution can not serve as the solution of the given original equation

$\therefore x = n\pi + \frac{\pi}{4}$ is the only possible solution.

Example :13 If $0 \leq x \leq 2\pi$, then the number of solutions of equation $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is

(1) 0 (2) 1 (3) 2 (4) 4

Solution : $3(\sin x + \cos x) - 2(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x) = 8$

$\Rightarrow (\sin x + \cos x)[3 - 2 + 2\sin x \cos x] = 8$

$\Rightarrow (\sin x + \cos x)[1 + 2 \sin x \cos x] = 8$

$\Rightarrow (\sin x + \cos x)^3 = 8$, $\sin x + \cos x = 2$

No solution

Example: 14 Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all the value of θ is of the term $k\pi$ then value of k is

(1) 6 (2) 5 (3) 4 (4) 3

Solution : Since, L.H.S ≥ 6 and R.H.S = 6, so equality holds only if $\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore$ sum $= 5\pi \Rightarrow k = 5$

Example: 15 If $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$ and $n \in I$, then

(1) $p + q = 0$ (2) $p + q = 2n + 1$ (3) $p + q = 2n$ (4) $p + q = 2(2n + 1)$

Solution : $\tan\left(\frac{p\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{q\pi}{4}\right)$

$\Rightarrow \frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4} \Rightarrow \frac{(p+q)}{4} = n + \frac{1}{2}$

$\Rightarrow (p + q) = 2(2n + 1)$

Example: 16 The number of solution of the equation $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $0 \le x \le 2\pi$ is

(1) 7 (2) 5 (3) 4 (4) 6

Solution : $(\sin x + \sin 3x) + \sin 2x = (\cos x + \cos 3x) + \cos 2x$

$\Rightarrow 2\sin 2x \cos x + \sin 2x = 2\cos 2x \cos x + \cos 2x$

$\Rightarrow \sin 2x\,(2\cos x + 2) = \cos 2x\,(2\cos x + 1)$

$\Rightarrow \cos x = -\frac{1}{2}, \tan 2x = 1$

$x = 2n\pi \pm \frac{2\pi}{3}, \frac{n\pi}{2} + \frac{\pi}{8}$

$0 \le x \le 2\pi$

$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{9\pi}{8}, \frac{4\pi}{3}, \frac{13\pi}{8}$

Example :17 The general solution of the equation $\sin 3\alpha = 4 \sin\alpha \sin(x + \alpha) \sin(x - \alpha)$ is $(\alpha \ne n\pi)$.

(1) $n\pi \pm \frac{\pi}{4}\ \forall n \in I$ (2) $n\pi \pm \frac{\pi}{3}\ \forall n \in I$

(3) $n\pi \pm \frac{\pi}{9}\ \forall n \in I$ (4) $n\pi \pm \frac{\pi}{12}\ \forall n \in I$

Solution : $\sin 3\alpha = 4 \sin\alpha\,(\sin^2 x - \sin^2\alpha)$

$\Rightarrow 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha \sin^2 x - 4\sin^3\alpha$

$\Rightarrow 3\sin\alpha = 4\sin\alpha \sin^2 x \Rightarrow \sin x\,(3 - 4\sin^2 x) = 0$

If $\sin\alpha = 0$; i.e. $\alpha = n\pi$, then equation becomes an identity.

$\sin^2 x = \frac{3}{4}, \sin x = \pm\frac{\sqrt{3}}{2}$

$x = n\pi \pm \frac{\pi}{3}\ \forall n \in I$

Example: 18 Find the general value of θ satisfying both $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ simultaneously.

Solution : $\sin\theta < 0$, $\tan\theta > 0$, so θ is in third quadrant.

Now, $\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}$

Common solution = $\frac{7\pi}{6}$

So general solution $\theta = 2n\pi + \frac{7\pi}{6}$.

Example: 19 Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ in $[0, 2\pi]$.

Solution : $(1 + \cos x)(2\sin x - \cos x) - (1 - \cos^2 x) = 0$

$(1 + \cos x)[(2\sin x - \cos x) - (1 - \cos x)] = 0, \quad (1 + \cos x)(2\sin x - 1) = 0$

$\sin x = \frac{1}{2}, \cos x = -1.$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi.$

Example: 20 Find the general solutions of equation $1 - \cos x \cot x = \cot x - \cos x$.

Solution : $(1 + \cos x)(1 - \cot x) = 0 \Rightarrow \cos x = -1$ or $\cot x = 1$

$\Rightarrow (2n + 1)\pi, \quad n\pi + \frac{\pi}{4}, \quad n \in I$

Example: 21 Find the general solution of $\sin^2 x + \frac{1}{4}\sin^2 3x = \sin x \sin^2 3x$

Solution : $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4}\sin^2 3x = 0$

$\Rightarrow \left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4}\sin^2 3x(1 - \sin^2 3x) = 0$

$\Rightarrow \left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4}\sin^2 3x \cos^2 3x = 0$

$\Rightarrow \left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{16}\sin^2 6x = 0$

$\Rightarrow \sin x - \frac{1}{2}\sin^2 3x = 0$ and $\sin 6x = 0$

$\Rightarrow 2\sin x = \sin^2 3x$ and $\sin 6x = 0 \Rightarrow x = \frac{k\pi}{6}\ k \in I.$

Now, $\sin^2\left(3\left(\frac{k\pi}{6}\right)\right) = \sin^2\left(\frac{k\pi}{2}\right) = \begin{cases} 1, \text{if } k \text{ is odd} \\ 0, \text{if } k \text{ is even} \end{cases}$

$\Rightarrow \sin x = 0$ or $\frac{1}{2}$

$\Rightarrow x = n\pi$ or $x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

Example: 22 Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$?

Solution : $\sin(1-x) \geq 0$ and $\cos x \geq 0$

$\sin(1-x) = \cos x, \cos\left(\frac{\pi}{2} - (1-x)\right) = \cos x$

$\Rightarrow \frac{\pi}{2} - 1 + x = 2n\pi \pm x \Rightarrow x = \frac{2n\pi - \frac{\pi}{2} + 1}{2}$

For n = 2, $x=\frac{7\pi}{4}+\frac{1}{2}$ which is the smallest positive root of the given equation.

Example: 23 Solve the inequality $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right)>\frac{1}{2}$

Solution : $1-2\sin^2\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right)>\frac{1}{2}$

$\Rightarrow 1-\frac{1}{2}\sin^2\left(\frac{2x}{3}\right)>\frac{1}{2} \Rightarrow \sin^2\left(\frac{2x}{3}\right)<1$

which is always true except when $\sin^2\left(\frac{2x}{3}\right)=1$

$\frac{2x}{3}=n\pi\pm\frac{\pi}{2}$ or $x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}$

So, solution of x is R~ {x: $x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}$ n, ∈ I}

Example :24 Solve 3tan 2x -4tan 3x = $\tan^2 3x$ tan 2x.

Solution : 3 (tan 2x- tan 3x) = tan 3x (1+tan3x tan 2x)

$\Rightarrow 3\left(\frac{\tan 2x-\tan 3x}{1+\tan 3x\tan 2x}\right)=\tan 3x \Rightarrow -3\tan(3x-2x)=\tan 3x$

$\Rightarrow -3\tan x=\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)\Rightarrow -3=\frac{3-\tan^2 x}{1-3\tan^2 x}$

tan x = 0 so x = $m\pi$

or $\tan^2 x=\frac{3}{5}$ so $x=n\pi\pm\tan^{-1}\sqrt{\frac{3}{5}}$, $n\in I$