Chapter 01

Vector

CONTENT

1. INTRODUCTION:

A vector is a quantity which has both the magnitude and the direction. Geometrically a vector is represented by a directed line segment.

2. SCALAR AND VECTOR QUANTITIES

A physical quantity which is completely specified by its magnitude only is called **scalar**. It is represented by a real number along with suitable unit.

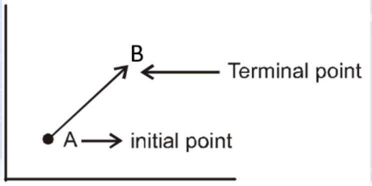
For example, Distance, Mass, Length, Time, Volume, Speed, Area are scalars.

On the other hand, a physical quantity which has magnitude as well as direction is called a **vector.** For example, Displacement, velocity, acceleration, force etc. are vector quantities.

3. REPRESENTATION OF A VECTOR

Geometrically a vector is represented by a directed line segment. If for a vector $\vec{a}$, $\vec{a} = \overrightarrow{AB}$, then **A** is called its **initial point** and **B** is called its **terminal point**. Clearly $\overrightarrow{AB}$ and $\overrightarrow{BA}$ represents different line segments

If $\vec{a} = \overrightarrow{AB}$, then its magnitude is expressed by $|\vec{a}|$ or $|\overrightarrow{AB}|$ or AB.



DETECTIVE MIND

- For a vector $\overrightarrow{AB} = a\hat{j} + b\hat{j} + c\hat{k}$

$$\left|\overrightarrow{AB}\right| = \sqrt{a^2 + b^2 + c^2}$$

- The magnitude of a vector is always non negative real number.

4. KINDS OF VECTORS

4.1 Zero or null vector :

A vector whose magnitude is zero is called **zero or null vector** and it is denoted by **0** or $\vec{0}$. The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

4.2 Unit vector :

A vector of unit magnitude is called a **unit vector**. A unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. Thus

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{\text{magnitude of } \vec{a}}$$

DETECTIVE MIND

- $|\hat{a}| = 1$
- Unit vectors parallel to x-axis, y-axis and z-axis are denoted by i, j and k respectively.
- Two unit vectors may not be equal unless they have the same direction.

4.3 Equal Vector :

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal , if

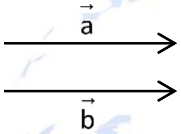
(1) $|\vec{a}| = |\vec{b}|$

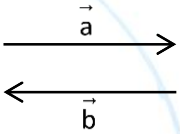
(2) they have the same direction

4.4 Collinear vectors:

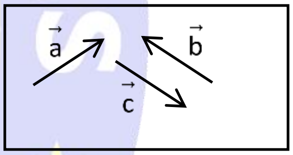
Vectors which are parallel to the same vectors and have either terminal or initial point in common are called collinear vectors.

4.5 Parallel Vectors :- Vectors having same line of support. The angle betweenw them is O° or 180°.





4.6 Coplanar Vectors :



If the directed line segment of some given vectors lie in a plane then they are called **coplanar vectors**. It should be noted that two vectors having the same initial point are always coplanar but such three or more vectors may not be coplanar.

4.7 Position Vectors :

The vector $\overrightarrow{OA}$ which represents the position of the point A with respect to a fixed point (called origin) O is called position vector of the point A. If (x,y,z) are coordinates of the point A, then

$$\overrightarrow{OA} = x\hat{i} + y\hat{j} + z\hat{k}$$

4.8 Reciprocal vectors :

A vector which has the same direction as vector 'a' but whose magnitude is the reciprocal of the magnitude of 'a', is called the reciprocal vector of vector 'a' and is denoted by $(\vec{a})^{-1}$.

Thus if $a = \alpha\,\hat{a}$, then

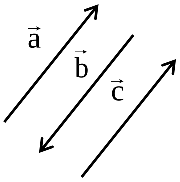
$$(\vec{a})^{-1} = \frac{1}{\alpha}\,.\,\hat{a} = \frac{\alpha\hat{a}}{|\vec{a}|^2} = \frac{\vec{a}}{|\vec{a}|^2}$$

DETECTIVE MIND

A unit vector is self reciprocal.

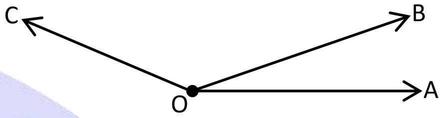
4.9 Like and Unlike vector :

The parallel vectors having the same direction are known as like vectors. On the contrary, the vectors having the opposite direction with respect to each other are termed as unlike vectors.



In the diagram $\vec{a}$ and $\vec{c}$ are like vectors whereas $\vec{a}$ and $\vec{b}$ are unlike vectors.

4.10 Co - initial vectors :



If two or more vectors have the same starting point then, they are said to be a co-initial vector for example, Vectors $\overrightarrow{OC}$, $\overrightarrow{OB}$ and $\overrightarrow{OA}$ are called co-initial vectors because they have the same starting point O.

4.11 Negative vector :

A negative of a vector represents the direction opposite to the reference direction. It means that the magnitude of two vectors are same but they are opposite in direction.

For example, if A and B are two vectors that have equal magnitude but opposite in direction, then vector A is negative of vector B.

A = – B

Example: Find unit vector of $\hat{i}-2\hat{j}+3\hat{k}$

Solution : $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then $|\vec{a}| = \sqrt{a_x^2+a_y^2+a_z^2}$

$\therefore \quad |\vec{a}| = \sqrt{14}$

$$\Rightarrow \quad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Example: Find values of x & y for which the vectors

$\vec{a} = (x + 2)\ \hat{i} - (x - y)\ \hat{j} + \hat{k}$

$\vec{b} = (x - 1)\ \hat{i} + (2x + y)\ \hat{j} + 2\hat{k}$ are parallel.

Solution: $\vec{a}$ and $\vec{b}$ are parallel if $\frac{x+2}{x-1} = \frac{y-x}{2x+y} = \frac{1}{2}$

x = – 5, y = – 20

5. ADDITION OF VECTORS

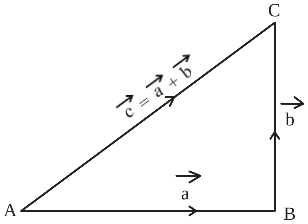
5.1 Addition in component form :

If the vectors are defined in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$. i.e. if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then their sum is defined as

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

5.2 Triangle law of addition :

If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the **triangle law of addition of vectors**.

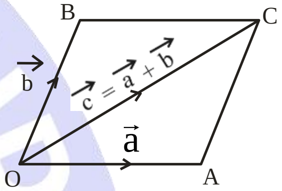


Thus, if $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, and $\overrightarrow{AC} = \vec{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

5.3 Parallelogram Law of Addition :

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as **parallelogram law of addition of vectors.**



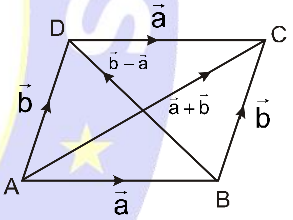
Thus if $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$

then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

Where $\overrightarrow{OC}$ is a diagonal of the parallelogram OACB.

Example: If $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Solution: Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.



Then, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow$ $\overrightarrow{AC} = \vec{a} + \vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$|\overrightarrow{AC}| = \sqrt{9+16+25} = \sqrt{50}$

$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\Rightarrow$ $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \vec{b} - \vec{a} = -(\hat{i} + 2\hat{j} + 3\hat{k})$

$|\overrightarrow{BD}| = \sqrt{1+4+9} = \sqrt{14}$

$\therefore$ Unit vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{\sqrt{50}}\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right)$

and Unit vector along $\overrightarrow{BD} = \dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = -\dfrac{1}{\sqrt{14}}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

6. PROPERTIES OF VECTOR ADDITION

Vector addition has the following properties.

(i) **Binary Operation :** The sum of two vectors is always a vector.

(ii) **Commutativity :**

For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(iii) **Associativity :**

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

(iv) **Identity :**

zero vector is the identity for addition for any vector $\vec{a}$

$\vec{0} + \vec{a} = \vec{a} = \vec{a} + \vec{0}$

(v) **Additive inverse :**

For every vector its negative vector $-\vec{a}$ exists. such that

$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

i.e. $(\vec{a})$ is the additive inverse of the vector $\vec{a}$.

Also if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then $-\vec{a} = -a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$

(vi) **Cancellation Law :**

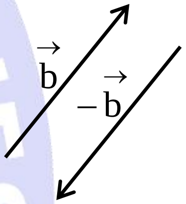
For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\left.\begin{array}{l}\vec{a}+\vec{b}=\vec{a}+\vec{c}\\ \vec{b}+\vec{a}=\vec{c}+\vec{a}\end{array}\right\} \Rightarrow \vec{b} = \vec{c}$$

6. SUBTRACTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two vectors, then their subtraction $\vec{a} - \vec{b}$ is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

where $-\vec{b}$ is the negative of $\vec{b}$ having magnitude equal to that of $\vec{b}$ and direction opposite to $\vec{b}$.



If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} - \vec{b} = (\vec{a}_1 - \vec{b}_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

DETECTIVE MIND

- $\vec{a} - \vec{b} \neq \vec{b} - \vec{a}$
- $(\vec{a} - \vec{b}) - \vec{c} \neq \vec{a} - (\vec{b} - \vec{c})$
- Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors $\vec{a}$ and $\vec{b}$, we have

  $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

  $|\vec{a} + \vec{b}| \geq |\vec{a}| - |\vec{b}|$

  $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$

  $|\vec{a} - \vec{b}| \geq |\vec{a}| - |\vec{b}|$

8. DISTANCE BETWEEN TWO POINTS

Let A and B be two given points whose coordinate are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

If $\vec{a}$ and $\vec{b}$ are position vector of A and B relative to point O, then

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Distance between the points = magnitude of $\overrightarrow{AB} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Example: Show by using distance formula that the points (4, 5, –5), (0, –11, 3) and (2, –3,–1) are collinear.

Solution: Let $A \equiv (4, 5, -5)$, $B \equiv (0, -11, 3)$, $C \equiv (2, -3, -1)$.

$$AB = \sqrt{(4-0)^2+(5+11)^2+(-5-3)^2} = \sqrt{336} = \sqrt{4\times 84} = 2\sqrt{84}$$

$$BC = \sqrt{(0-2)^2+(-11+3)^2+(3+1)^2} = \sqrt{84}$$

$$AC = \sqrt{(4-2)^2+(5+3)^2+(-5+1)^2} = \sqrt{84}$$

$$BC + AC = AB$$

Hence points A, B, C are collinear and C lies between A and B.

Example: Find the locus of a point which moves such that the sum of its distances from points A $(0, 0, -\alpha)$ and B$(0, 0, \alpha)$ is constant.

Solution: Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$$\therefore \quad \sqrt{(x-0)^2+(y-0)^2+(z+\alpha)^2} + \sqrt{(x-0)^2+(y-0)^2+(z-\alpha)^2} = 2a$$

$$\Rightarrow \quad \sqrt{x^2+y^2+(z+\alpha)^2} = 2a - \sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow \quad 4z\alpha - 4a^2 = -4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow \quad \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$$

$$\text{or,} x^2 + y^2 + z^2\left(1-\frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2$$

$$\Rightarrow \quad \frac{x^2}{a^2-\alpha^2} + \frac{y^2}{a^2-\alpha^2} + \frac{z^2}{a^2} = 1$$

This is the required locus.

9. MULTIPLICATION OF A VECTOR BY A SCALAR

If $\vec{a}$ is a vector and m is a scalar (i.e. a real number) then $m\vec{a}$ is a vector whose magnitude is m times that of s $\vec{a}$ and whose direction is the same as that of $\vec{a}$, if m is positive then direction of $m\vec{a}$ is same as that of $\vec{a}$ and if m is negative, then direction is opposite to that of $\vec{a}$.

$\therefore$ magnitude of $m\vec{a} = |m\vec{a}|$

$\Rightarrow$ m (magnitude of a) = $m|\vec{a}|$

Again if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

$m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$

DETECTIVE MIND

- The multiplication of a vector by a scalar is also named as ‘scalar multiplication’.
- From the definition of scalar multiplication it is obvious to note that

  $\vec{a} \,||\, \vec{b} \Rightarrow \vec{a} = m\vec{b}$, where m is some suitable scalar.

Properties

If $\vec{a}, \vec{b}$ are any two vectors and m, n are any scalar then

(i) $m(\vec{a}) = (\vec{a})\, m = m\vec{a}$ (commutativity)

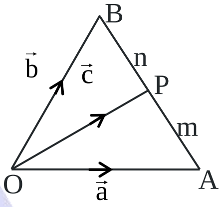
(ii) $m\,(n\vec{a}) = n\,(m\vec{a}) = (mn)\,\vec{a}$ (Associativity)

(iii) $(m + n)\,\vec{a} = m\vec{a} + n\vec{a}$

(iv) $m\,(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

(Distributivity)

10. POSITION VECTOR OF A DIVIDING POINT OR SECTION FORMULA



If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector c of a point P dividing AB in the ratio m : n is given by

$$\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

Particular Case :

(i) Position vector of the mid point of AB is $\dfrac{\vec{a}+\vec{b}}{2}$

(ii) Any vector along the internal bisector of $\angle AOB$ is given by

$\lambda(\vec{a} + \vec{b})$

DETECTIVE MIND

- ➢ If the point P divides AB in the ratio m: n externally, then m/n will be negative. If m is positive and n is negative, then position vector of P is given by $\vec{c} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$
- ➢ If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a triangle, then position vector of its centroid is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$
- ➢ If a,b,c,d are position vectors of vertices of a tetrahedron, then position vector of its centroid is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

Example: The midpoint of two opposite sides of quadrilateral and the midpoint of the diagonals are vertices of a parallelogram. Prove using vectors.

Solution: Let ABCD be a quadrilateral and $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of vertices A, B, C, D respectively.

Let E, F, G, H be midpoint of AB, CD, AC and BD respectively

position vector of E = $\dfrac{\vec{a}+\vec{b}}{2}$

position vector of F = $\dfrac{\vec{c}+\vec{d}}{2}$

position vector of G = $\dfrac{\vec{a}+\vec{c}}{2}$

position vector of H = $\frac{\vec{b}+\vec{d}}{2}$

$$\overrightarrow{EG} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{\vec{c}-\vec{b}}{2}$$

$$\overrightarrow{HF} = \frac{\vec{c}+\vec{d}}{2} - \left(\frac{\vec{b}+\vec{d}}{2}\right) = \frac{\vec{c}-\vec{b}}{2}$$

$\overrightarrow{EG} = \overrightarrow{HF} \Rightarrow$ $\overrightarrow{EG} || \overrightarrow{HF}$ and EG = HF

hence EGFH is a parallelogram.

11. RELATION BETWEEN TWO PARALLEL VECTORS

(i) If $\vec{a}$ and $\vec{b}$ be two parallel vectors, then there exists a scalar k such that $\vec{a} = k\vec{b}$

i.e. there exist two non- zero scalar quantities x and y so that $x\vec{a} + y\vec{b} = 0$

If a and b be two non-zero non-parallel vectors then $x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0$ and $y = 0$ obviously

$$x\vec{a} + y\vec{b} = 0 \Rightarrow \begin{cases} \vec{a} = 0, \vec{b} = 0 \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \vec{a} || \vec{b} \end{cases}$$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then from the property of parallel vector, we have

$\vec{a} \ || \ \vec{b}$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

12. COLLINEARITY OF THREE POINTS

(i) If $\vec{a}, \vec{b}, \vec{c}$ be position vectors of three points A,B and C respectively and x, y,z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that

$x\vec{a} + y\vec{b} + z\vec{c} = 0$ and $x + y + z = 0$

(ii) Three points A, B and C are collinear, if any two vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are parallel i.e. one of them is scalar multiple of any one of the remaining vectors.

13. COPLANAR & NON-COPLANAR VECTOR

(i) If $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors, then a vector $\vec{c}$ can be expressed uniquely as linear combination of remaining two vectors i.e.

$$\vec{c} = \lambda\vec{a} + \mu\vec{b}$$

Where λ and μ are suitable scalars.

Again $\vec{c} = \lambda\vec{a} + \mu\vec{b} \Rightarrow$ vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

If $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors, then there exist three non zero scalars x, y, z so that

$$x\vec{a} + y\vec{b} + z\vec{c} = 0$$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ be three non coplanar non zero vector then

$$x\vec{a} + y\vec{b} + z\vec{c} = 0 \Rightarrow x = 0, y = 0, z = 0$$

(iii) Any vector r can be expressed uniquely as the linear combination of three non coplanar and non- zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $r = x\vec{a} + y\vec{b} + z\vec{c}$ where x, y and z are scalars.

14. PRODUCT OF VECTORS

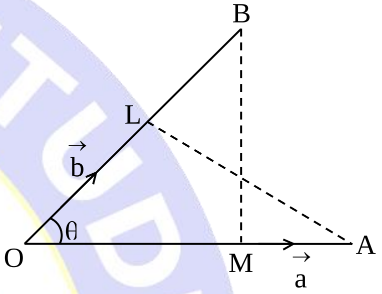
Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called **scalar product**. It is also called as **dot product** because this product is represented by putting a dot. When the product of two vectors results in a vector quantity then this product is called **Vector Product**. This product is represented by (x) sign so that it is also called as **cross product.**

15. SCALAR OR DOT PRODUCT OF TWO VECTORS

15.1 Definition :

If $\vec{a}$ and $\vec{b}$ are two non zero vectors and θ be the angle between them, then their **scalar product** (or dot product) is defined as the number $|\vec{a}|.\ |\vec{b}|\ \cos\theta$ where $|\vec{a}|$ and $|\vec{b}|$ are modulii of $\vec{a}$ and $\vec{b}$ respectively and $0 \le \theta \le \pi$. It is denoted by $\vec{a}\,.\,\vec{b}$. Thus

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = a\,b\,\cos\theta$$



DETECTIVE MIND

- ➢ $\vec{a}\,.\,\vec{b} \in R$
- ➢ $\vec{a}\,.\,\vec{b} \le |\vec{a}|\,|\vec{b}|$
- ➢ $\vec{a}\,.\,\vec{b} > 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ is acute $\vec{a}\,.\,\vec{b} < 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ is obtuse.
- ➢ The dot product of a zero and non- zero vector is a scalar zero.

15.2 Geometrical Interpretation :

Geometrically, the scalar product of two vectors is equal to the product of the magnitude of one and the projection of second in the direction of first vector i.e. $\vec{a}\,.\,\vec{b} = \vec{a}\,(\vec{b}\,\cos\theta)$

$= |\vec{a}|$ (projection of $\vec{b}$ in the direction of $\vec{a}$)

Similarly $\vec{a}\,.\,\vec{b} = \vec{b}\,(\vec{a}\,\cos\theta) = |\vec{b}|$ (projection of $\vec{a}$ in the direction of $\vec{b}$)

Here projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

16. SCALAR PRODUCT IN PARTICULAR CASES

(i) If $\vec{a}$ and $\vec{b}$ are like vectors, then $\theta = 0$, so $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}| = a\,b$ i.e. scalar product of two like vectors is equal to the product of their modulii.

(ii) If $\vec{a}$ and $\vec{b}$ are unlike vectors then $\theta = \pi$, so $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\pi = -\,a\,b$.

(iii) The scalar product of a vector by itself is equal to the square of its modulus i.e.

$$\vec{a}\,.\,\vec{a} = |\vec{a}|^2$$

(iv) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then $\theta = \pi/2$, so $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos \pi/2 = 0$ i.e. the scalar product of two perpendicular vectors is always zero.

But its converse may not be true i.e. $\vec{a}\,.\,\vec{b} = 0 \nRightarrow \vec{a} \perp \vec{b}$

But if a and b are non zero vectors, then $\vec{a}\,.\,\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

Thus $\vec{a} \neq 0,\ \vec{b} \neq 0,\ \vec{a}\,.\,\vec{b} = 0, \Rightarrow \vec{a} \perp \vec{b}$

(v) With the help of the above cases, we get the following important results:

(1) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ (2) $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) If $\vec{a}$ and $\vec{b}$ are unit vectors, then $\vec{a}\,.\,\vec{b} = \cos\theta$

17. PROPERTIES OF SCALAR PRODUCT

If $\vec{a}, \vec{b}, \vec{c}$ are any vectors and m and n are any scalars then

(i) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ (Commutativity)

(ii) $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$

(iii) $(m\vec{a}).(n\vec{b}) = (mn)(\vec{a}.\vec{b})$

(iv) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributivity)

(v) $\vec{a}.\vec{b} = \vec{a}.\vec{c} \nRightarrow \vec{b} = \vec{c}$

Infact $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

(vi) $(\vec{a}.\vec{b}).\vec{c}$ is meaningless

(vi) scalar product is not binary operation.

DETECTIVE MIND

- ➢ $(\vec{a}.\vec{b}).\vec{b}$ is not defined
- ➢ $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a}.\vec{b} + |\vec{b}|^2$
- ➢ $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2\,\vec{a}.\vec{b} + |\vec{b}|^2$
- ➢ $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$
- ➢ $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow \vec{a} \,||\, \vec{b}$
- ➢ $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Rightarrow \vec{a} \perp \vec{b}$
- ➢ $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \Rightarrow \vec{a} \perp \vec{b}$

18. ANGLE BETWEEN TWO VECTORS

(i) If $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}}{|\vec{a}|}.\frac{\vec{b}}{|\vec{b}|} = \hat{a}.\hat{b}$$

(ii) If $\vec{a} = \vec{a}_1\,\hat{i} + \vec{a}_2\,\hat{j} + \vec{a}_3\,\hat{k}$ and

$\vec{b} = \vec{b}_1\,\hat{i} + \vec{b}_2\,\hat{j} + \vec{b}_3\,\hat{k}$ then

$$\cos\theta = \frac{\vec{a}_1\vec{b}_1 + \vec{a}_2\vec{b}_2 + \vec{a}_3\vec{b}_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

DETECTIVE MIND

- If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

Example: If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 6$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution: We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow \quad \left(\vec{a} + \vec{b}\right) . \left(\vec{a} + \vec{b}\right) = \left(-\vec{c}\right) . \left(-\vec{c}\right)$

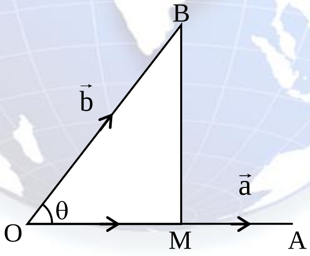
$\Rightarrow \quad \left|\vec{a} + \vec{b}\right|^2 = |\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2 + \left|\vec{b}\right|^2 + 2\,|\vec{a}|\left|\vec{b}\right| \cos\theta = |\vec{c}|^2$

$\Rightarrow \quad 9 + 25 + 2\,(3)\,(5) \cos\theta = 36$

$\Rightarrow \quad \cos\theta = \frac{2}{30} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{1}{15}$

19. COMPONENTS OF b ALONG & PERPENDICULAR TO a



(i) Component along a = $\overrightarrow{OM}$ = OM $\hat{a}$ = (b cos θ) $\hat{a}$ = $\frac{(ab\cos\theta)}{a}\,\hat{a} = \frac{(\vec{a}.\vec{b})}{\left|\vec{a}\right|^2}\,.\,\vec{a}$

(ii) Component perpendicular to $\vec{a} = \overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OM} = \vec{b} - \frac{(\vec{a}.\vec{b})}{\left|\vec{a}\right|^2}.(\vec{a})$

Example: If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$.

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but ⊥ to $\vec{a}$.

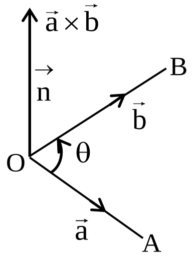
Solution: (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$; Here $\vec{a}.\vec{b} = 2 - 1 + 3 = 4$ and $|\vec{a}|^2 = 3$

Hence $\left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{4}{3}\vec{a} = \frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but ⊥ to $\vec{a}$ is $\vec{b} - \left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

20. VECTOR OR CROSS PRODUCT OF TWO VECTORS

20.1 Definition :



If $\vec{a}$ and $\vec{b}$ be two vectors and θ $(0 \le \theta \le \pi)$ be the angle between them, then their vector (or cross) product is defined to be a vector whose magnitude is ab sin θ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ form a right handed system.

$\therefore$ $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n} = \vec{a}\,\vec{b}\,\sin\theta\,\hat{n}$

Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

20.2 Vector product in terms of components :

If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b} = (a_2 b_3 - a_3 b_2)\,\hat{i} + (a_3 b_1 - a_1 b_3)\,\hat{j}$
$+ (a_1 b_2 - a_2 b_1)\,\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

20.3 Angle between two vectors :

If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

If $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, then $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

21. VECTOR PRODUCT IN PARTICULAR CASES

(i) The vector product of two parallel vectors is always zero i.e. if vectors a and b are parallel, then $\vec{a}\times\vec{b} = 0$

In particular $\vec{a}\times\vec{b} = 0$

(ii) If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\,\hat{n} = ab\,\hat{n}$

(iii) If $\hat{i},\hat{j},\hat{k}$ be three mutually perpendicular unit vectors, then

(1) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(2) $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(3) $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$

22. PROPERTIES OF VECTOR PRODUCT

If $\vec{a}, \vec{b}, \vec{c}$ are any vectors and m and n are any scalars then

(i) $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ (Non- commutativity)

but $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

and $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{a}|$

(ii) $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$

(iii) $(m\vec{a}) \times (n\vec{b}) = (mn)(\vec{a} \times \vec{b})$

(iv) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(v) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$
(Distributivity)

(vi) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \not\Rightarrow \vec{b} = \vec{c}$ Infact

$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \,||\, (\vec{b} - \vec{c})$

Example: Find a vector of magnitude 9, which is perpendicular to both the vectors $\hat{i} - 7\hat{j} + 7\hat{k}$ and $3\hat{i} - 2\hat{j} + 2\hat{k}$

Solution: Let $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$. Then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = (-14 + 14)\hat{i} - (2 - 21)\hat{j} + (-2 + 21)\hat{k} = 19\hat{j} + 19\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = 19\sqrt{2}$$

$$\therefore \quad \text{Required vector} = \pm 9\left(\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}\right) = \pm\frac{9}{\sqrt{2}}(\hat{j} + \hat{k})$$

Example: For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

Solution: We have, $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$ [Using distributive law]

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c} = \vec{0}$ [$\because \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ etc.]

23. GEOMETRICAL INTERPRETATION OF VECTOR PRODUCT

The vector product of the vectors $\vec{a}$ and $\vec{b}$ represents a vector whose modulus is equal to the area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$. Therefore $|\vec{a} \times \vec{b}|$ = area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

Further it should be noted that if $\vec{a}$, $\vec{b}$ represent two diagonals of a parallelogram, then the area of the parallelogram $= \frac{1}{2}|\vec{a} \times \vec{b}|$

DETECTIVE MIND

- Area of a quadrilateral ABCD $= \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$

24. AREA OF A TRIANGLE

(i) Area of triangle ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a ΔABC then its

$$\text{Area} = \frac{1}{2}\,|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$$

DETECTIVE MIND

- ➢ Three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear if $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = 0$

25. SCALAR TRIPLE PRODUCT

25.1 Definition :

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors $\vec{a}$ and $\vec{b} \times \vec{c}$. It is generally denoted by $\vec{a}\,.\,(\vec{b} \times \vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}\,]$. It is read as box product of $\vec{a}, \vec{b}, \vec{c}$. Similarly other scalar triple products can be defined as $(\vec{b} \times \vec{c}).\vec{a}$, $(\vec{c} \times \vec{a}).\vec{b}$.

DETECTIVE MIND

- ➢ Scalar triple product always results in a scalar quantity (number).

25.2 Geometrical Interpretation :

The scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vector.

Therefore $(\vec{a} \times \vec{b}).\ \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}\,]$ = Volume of the parallelopiped whose coterminous edges are $\vec{a}, \vec{b}$ and $\vec{c}$

25.3 Formula for scalar Triple Product :

(i) If $\vec{a} = a_1\ell + a_2m + a_3n$, $\vec{b} = b_1\ell + b_2m + b_3\,n$ and $\vec{c} = c_1\ell + c_2m + c_3n$, then

$$[\vec{a}\ \vec{b}\ \vec{c}\,] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\ell mn]$$

(ii) $\vec{a} = \vec{a}_1\,\hat{i} + \vec{a}_2\,\hat{j} + \vec{a}_3\,\hat{k}$, $\vec{b} = \vec{b}_1\,\hat{i} + \vec{b}_2\,\hat{j} + \vec{b}_3\,\hat{k}$ and $\vec{c} = \vec{c}_1\,\hat{i} + \vec{c}_2\,\hat{j} + \vec{c}_3\,\hat{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}\,] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

(1) $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}\,] = 2\,[\vec{a}\ \vec{b}\ \vec{c}\,]$

(2) $[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}\,] = 0$

(3) $[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\,] = 2\,[\vec{a}\ \vec{b}\ \vec{c}\,]^2$

25.4 Properties of Scalar Triple product

(i) The position of (.) and (×) can be interchanged i.e. $\vec{a}\,.\,(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\,.\,\vec{c}$

but $(\vec{a}\times\vec{b})\,.\,\vec{c} = \vec{c}\,.\,(\vec{a}\times\vec{b})$

So $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

Therefore if we don't change the cyclic order of a, b and c then the value of scalar triple product is not changed by interchanging dot and cross.

(ii) If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$\vec{a}\,.\,[\vec{b}\times\vec{c}] = -\,\vec{a}\,.(\vec{c}\times\vec{b})$ or $[\vec{a}\ \vec{b}\ \vec{c}] = -\,[\vec{a}\ \vec{c}\ \vec{b}]$

from (i) and (ii) we have

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -\,[\vec{a}\ \vec{c}\ \vec{b}] = -\,[\vec{b}\ \vec{a}\ \vec{c}] = -\,[\vec{c}\ \vec{b}\ \vec{a}]$

(iii) The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e.

$[\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = 0$

(iv) The scalar triple product of three mutually perpendicular unit vectors is ± 1. Thus

$[\hat{i}\ \hat{j}\ \hat{k}] = 1$, $[\hat{i}\ \hat{k}\ \hat{j}] = -1$

(v) If two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(vi) $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ i.e. the necessary and sufficient condition for three non-zero collinear vectors to be coplanar is

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(vii) For any vectors $\vec{a}, \vec{b}, \vec{c}, d$

$[\vec{a}+\vec{b}\ \ \vec{c}\ \ d] = [\vec{a}\ \vec{c}\ \ d] + [\vec{b}\ \ \vec{c}\ \ d]$

26. VECTOR TRIPLE PRODUCT

26.1 Definition :

The vector triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as the vector product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$.

26.2 Properties :

(i) Expansion formula for vector triple product is given by

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}$.

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{Then } \vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

DETECTIVE MIND

- Vector triple product is a vector quantity.
- $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$
- Vector $\vec{a}\times(\vec{b}\times\vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$
- The direction of $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and parallel to and $\vec{b}$ and $\vec{c}$.

27. LAGRANGE'S IDENTITY

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}]$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

This is called Lagrange's identity.

28. RECIPROCAL SYSTEM OF VECTORS

Two system of vectors are called reciprocal systems of vectors if by taking the dot product we get unity.

Thus, if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, and if

$\vec{a}\,' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b}\,' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}\,' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, then $\vec{a}\,',\vec{b}\,',\vec{c}\,'$ are the reciprocal systems of vectors for vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

PROPERTIES

I. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ and $\vec{a}\,'$, $\vec{b}\,'$ and $\vec{c}\,'$ are reciprocal system of vectors, then $\vec{a}\cdot\vec{a}\,' = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{a}\ \vec{b}\ \vec{c}} = \dfrac{[\vec{a}\vec{b}\vec{c}]}{\vec{a}\vec{b}\vec{c}} = 1$.

Similarly, $\vec{b}\ \vec{b}\,' = \vec{c}\cdot\vec{c}\,' = 1$.

II. $\vec{a}\cdot\vec{b}\,' = \vec{a}\cdot\vec{c}\,' = \vec{b}\cdot\vec{a}\,' = \vec{b}\cdot\vec{c}\,' = \vec{c}\cdot\vec{a}\,' = \vec{c}\cdot\vec{b}\,' = 0$

III. $[\vec{a}\vec{b}\vec{c}][\vec{a}\,'\vec{b}\,'\vec{c}\,'] = 1$

IV. The orthogonal triad of vectors $\hat{i},\hat{j}$ and $\hat{k}$ is self–reciprocal.

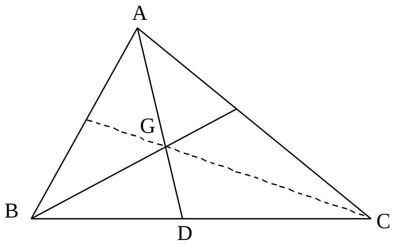
V. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non–coplanar iff $\vec{a}\,',\vec{b}\,'$ and $\vec{c}\,'$ are non–coplanars.

SOLVED EXAMPLES

Example: 1 If G is the centroid of triangle ABC then value of $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}$ will be-

(1) $\vec{0}$ (2) $3\overrightarrow{GA}$ (3) $3\overrightarrow{GB}$ (4) $3\overrightarrow{GC}$

Solution: If D is middle point of side BC then-



$\overrightarrow{GD} = \frac{1}{2}\ (\overrightarrow{GB} + \overrightarrow{GC}\,)$

$\because$ G divides AD in the ratio of 2 : 1

$\therefore\ \ \overrightarrow{AG} = 2\,\overrightarrow{GD}$

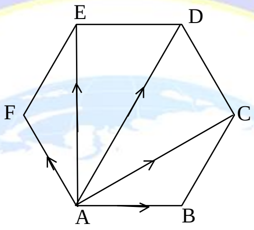
$\Rightarrow\ \ -\overrightarrow{GA} = \overrightarrow{GB} + \overrightarrow{GC}$

$\Rightarrow\ \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$

Example: 2 If ABCDEF is a regular hexagon and $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = k\ \overrightarrow{AD}$, then k equals-

(1) 2 (2) 3 (3) 6 (4) 5

Solution: $\because\ \overrightarrow{AB} = \overrightarrow{ED}$ and $\overrightarrow{AF} = \overrightarrow{CD}$,



so $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= \overrightarrow{ED} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{CD}$

$= (\overrightarrow{AC} + \overrightarrow{CD}) + (\overrightarrow{AE} + \overrightarrow{ED}\ ) + \overrightarrow{AD}$

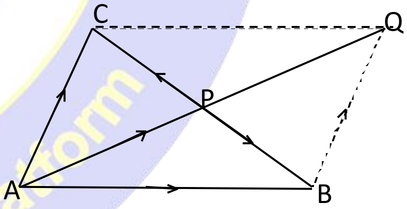
$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD} = 3\ \overrightarrow{AD}$

$\therefore$ k = 3

Example: 3 If a point P on the side BC of triangle ABC is such that $\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{CP} + \overrightarrow{PQ}$ then ABQC will be-

(1) Square (2) Rectangle (3) Parallelogram (4) None of these

Solution: From figure



$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AB}$

$\overrightarrow{CP} + \overrightarrow{PQ} = \overrightarrow{CQ}$

So $\overrightarrow{AB} = \overrightarrow{CQ}$ then ABQC is a parallelogram.

Example: 4 The length of diagonal AC of a parallellogram ABCD whose two adjacent sides AB and AD are represented respectively by $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$ is-

(1) 3 (2) 4 (3) 5 (4) 7

Solution: $\because\ \ \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

$\therefore$ Length of the diagonal $\overrightarrow{AC} = |\ \overrightarrow{AC}\ |$

$= \sqrt{3^2 + 6^2 + (-2)^2} = 7$

Example: 5 The position vector of the point A is $6\vec{b} - 2\vec{a}$ and the point P divides any line AB in the ratio 1: 2. If the position vector of P is $\vec{a} - \vec{b}$, then position vector of B will be-

(1) $7\vec{a}+15\vec{b}$ (2) $7\vec{a}-15\vec{b}$ (3) $15\vec{a}-7\vec{b}$ (4) $15\vec{a}+7\vec{b}$

Solution: Let the position vector of B be r

$$\therefore \quad \vec{a}-\vec{b} = \frac{\vec{r}+2(6\vec{b}-2\vec{a})}{1+2}$$

$$\Rightarrow \quad 3\vec{a}-3\vec{b} = \vec{r}+12\vec{b}-4\vec{a}$$

$$\Rightarrow \quad \vec{r} = 7\vec{a}-15\vec{b}$$

Example: 6 If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then A, B, C are-

(1) coplanar (2) collinear (3) non-collinear (4) None of these

Solution: $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC} \Rightarrow \overrightarrow{AB} = \overrightarrow{BC}$

$\therefore$ $\overrightarrow{AB} \,||\, \overrightarrow{BC}$ and a point B is common to both vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence A, B, C are collinear.

Example: 7 Let A = (x + 4y) $\vec{a}$ + (2x + y + 1) $\vec{b}$ and B = (y – 2x + 2) $\vec{a}$ + (2x – 3y – 1) $\vec{b}$ where $\vec{a}$ and $\vec{b}$ are non collinear vectors, if 3A = 2B; then

(1) x = 1, y = 2 (2) x = 2, y = 1 (3) x = 2, y = – 1 (4) x = – 1, y = 2

Solution: 3 A = 3(x + 4y) $\vec{a}$ + 3 (2x + y + 1) $\vec{b}$

2 B = 2(y – 2x + 2) $\vec{a}$ + 2 (2x – 3y – 1) $\vec{b}$

$\therefore$ 3A = 2B $\Rightarrow$ 3(x + 4y) = 2 (y – 2x + 2),

3 (2x + y + 1) = 2(2x – 3y – 1)

$\Rightarrow$ 7x + 10 y = 4 and 2x + 9y = – 5

$\Rightarrow$ x = 2, y = – 1

Example: 8 Let position vectors of points A, B, C and D are respectively $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$. If the points are coplanar, then the value of λ is-

(1) $-\frac{146}{17}$ (2) $\frac{146}{17}$ (3) 0 (4) None of these

Solution: $\overrightarrow{AB} = -\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$

& $\overrightarrow{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

If A, B, C, D are coplanar, then vectors

$\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then

$-\hat{i}+5\hat{j}-3\hat{k} = x\,(-4\hat{i}+3\hat{j}+3\hat{k})$

$+\, y\,[\hat{i}+7\hat{j}+(\lambda+1)\hat{k}]$

$\Rightarrow$ – 4x + y = – 1, 3x + 7y = 5

and 3x + (λ + 1)y = – 3

Solving first two equations

$$x = \frac{12}{31},\ y = \frac{17}{31}$$

Substituting these values of x and y in third equation, we get

$$\lambda = -\frac{146}{17}$$

Example: 9 If $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$, $\vec{b} = \hat{i}-3\hat{j}+2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+5\hat{k}$, then vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are -

(1) linearly independent (2) collinear

(3) linearly dependent (4) None of these

Solution: Let x ($\vec{a}$) + y($\vec{b}$) + z($\vec{c}$) = $\vec{0}$

$\Rightarrow$ x($\hat{i}+2\hat{j}-3\hat{k}$) + y($\hat{i}-3\hat{j}+2\hat{k}$) + z($2\hat{i}-\hat{j}+5\hat{k}$) = $\vec{0}$

$\Rightarrow$ $x + y + 2z = 0$

$2x - 3y - z = 0$

$-3x + 2y + 5z = 0$

Solving these equations, we get x = 0 = y = z

$\Rightarrow$ vectors are linearly independent.

Example: 10 If $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ are position vectors of vertices of a triangle, then it is-

(1) equilateral (2) isosceles

(3) right angled isosceles (4) None of these

Solution: If given points are A, B, C respectively, then

$|\overrightarrow{AB}| = |-\hat{i}-2\hat{j}-6\hat{k}| = \sqrt{1+4+36} = \sqrt{41}$

$|\overrightarrow{BC}| = |2\hat{i}-\hat{j}+\hat{k}| = \sqrt{4+1+1} = \sqrt{6}$

$|\overrightarrow{CA}| = |-\hat{i}+3\hat{j}+5\hat{k}| = \sqrt{1+9+25} = \sqrt{35}$

Now $|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

Hence given points form a right angled triangle.

Example: 11 If two vertices of a triangle are respectively $\hat{i}-\hat{j}$ and $\hat{j}+\hat{k}$, then the third vertex may be-

(1) $\hat{i}+\hat{k}$ (2) $\hat{i}-\hat{k}$ (3) $2\hat{i}-\hat{j}$ (4) All three

Solution: In the given alternatives no vector is collinear with the $\hat{i}-\hat{j}$ and $\hat{j}+\hat{k}$.

Example: 12 If $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$, then $|\vec{a}+\vec{b}|$ is equal to-

(1) $4\sqrt{6}$ (2) $3\sqrt{6}$ (3) $2\sqrt{6}$ (4) $\sqrt{6}$

Solution: $\vec{a}+\vec{b} = 3\hat{i}+3\hat{j}+6\hat{k}$

$\Rightarrow |\vec{a}+\vec{b}| = \sqrt{9+9+36} = \sqrt{54} = 3\sqrt{6}$

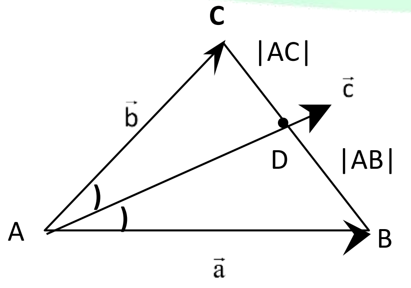
Example: 13 The vector $\vec{c}$, directed along the internal bisector of the angle between the vectors $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}| = 5\sqrt{6}$ is-

(1) $\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ (2) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (3) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (4) None of these

Solution: Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$

and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

Internal bisector divides the BC in the ratio of



$|\overrightarrow{AB}| : |\overrightarrow{AC}|$, $|\overrightarrow{AB}| = 9$, $|\overrightarrow{AC}| = 3$

$$\overrightarrow{AD} = \left(\frac{9\left(-2\hat{i}-\hat{j}+2\hat{k}\right)+3\left(7\hat{i}-4\hat{j}-4\hat{k}\right)}{9+3}\right)$$

$$\overrightarrow{AD} = \frac{\hat{i}-7\hat{j}+2\hat{k}}{4}$$

$$\vec{C} = \left(\frac{\overrightarrow{AD}}{|\overrightarrow{AD}|}\right)\left(5\sqrt{6}\right) = \frac{5}{3}\,(\hat{i}-7\hat{j}+2\hat{k})$$

Example: 14 The projection of the vector $\hat{i}+\hat{j}+\hat{k}$ on vector $\hat{i}+2\hat{j}+3\hat{k}$ is-

(1) $\frac{1}{\sqrt{14}}$ (2) $\frac{3}{\sqrt{14}}$ (3) $\frac{6}{\sqrt{14}}$ (4) $\frac{2}{3}$

Solution: Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ and so projection of $\vec{a}$ on $\vec{b}$ is

$$= \frac{(\hat{i}+\hat{j}+\hat{k}).(\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{1+4+9}} = \frac{1+2+3}{\sqrt{1+4+9}} = \frac{6}{\sqrt{14}}$$

Example:15 If for three vectors $\vec{a}, \vec{b}, \vec{c}$; $\vec{a}.\vec{b} = \vec{a}.\vec{c}$; then which of the following is correct-

(1) $\vec{b} = \vec{c}$ (2) $\vec{a} = \vec{0}$

(3) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$ (4) $\vec{a} \perp (\vec{b}-\vec{c})$

Solution: $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$\Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

Example: 16 If θ be the angle between vectors $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}+\hat{k}$, then cos θ equals-

(1) $\frac{5}{7}$ (2) $\frac{6}{7}$ (3) $\frac{4}{7}$ (4) $\frac{1}{2}$

Solution: $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{3+4+3}{\sqrt{14}\sqrt{14}} = 5/7$

Example:17 If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then angle between a and b is

(1) 60° (2) 30° (3) 90° (4) 180°

Solution: $|a + b| = |\vec{a}-\vec{b}|$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}.\vec{b}$

$\Rightarrow 4\,\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

Example: 18 Forces $3\hat{i}+2\hat{j}+5\hat{k}$ and $2\hat{i}+\hat{j}+3\hat{k}$ are acting at a particle which is displaced from point $2\hat{i}-\hat{j}-3\hat{k}$ to the point $4\hat{i}-3\hat{j}+\hat{k}$. The work done by forces is-

(1) 30 units (2) 36 units (3) 24 units (4) 18 units

Solution: Resultant force

$\vec{F} = (3\hat{i}+2\hat{j}+5\hat{k}) + (2\hat{i}+\hat{j}+3\hat{k})$

$= 5\hat{i}+3\hat{j}+8\hat{k}$

Displacement vector

$\vec{d} = (4\hat{i} - 3\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} - 3\hat{k}) = 2\hat{i} - 2\hat{j} + 4\hat{k}$

∴ work done by force = $\vec{F}\,.\,\vec{d}$

= 10 – 6 + 32 = 36 units

Example: 19 For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}\times b|^2$ equals-

(1) $|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$

(2) $|\vec{a}|^2 + |\vec{b}|^2$

(3) $|\vec{a}|^2 - |\vec{b}|^2$

(4) None of these

Solution: $|\vec{a}\times\vec{b}|^2 = (|\vec{a}|\ |\vec{b}|\ \sin\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}|\ |\vec{b}|\cos\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$

Example:20 If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that

$\vec{a}+\vec{b}+\vec{c} = \vec{0}$, then-

(1) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$

(2) $\vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(3) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(4) None of these

Solution: ∵ $\vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow \vec{c} = -(\vec{a}+\vec{b})$

∴ $\vec{b}\times\vec{c} = -\vec{b}\times(\vec{a}+\vec{b})$

$= -\vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$= \vec{a}\times\vec{b}$

Similarly $\vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

∴ $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

Example:21 If $\ell\hat{i}+m\hat{j}+n\hat{k}$ is a unit vector which is perpendicular to vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$ then $|\ell|$ is equal to-

(1) $-\dfrac{3}{\sqrt{155}}$

(2) $\sqrt{\dfrac{3}{155}}$

(3) $\dfrac{3}{\sqrt{155}}$

(4) None of these

Solution: Vector $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$

$$= \frac{\left(2\hat{i}-\hat{j}+\hat{k}\right)\times(3\hat{i}+4\hat{j}-\hat{k})}{|\left(2\hat{i}-\hat{j}+\hat{k}\right)\times(3\hat{i}+4\hat{j}-\hat{k})|}$$

$$= \frac{\hat{i}(1-4)-\hat{j}(-2-3)+\hat{k}(8+3)}{\sqrt{9+25+121}}$$

$$= \frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}$$

$$\therefore\ |\ell| = \left|\frac{-3}{\sqrt{155}}\right| = \frac{3}{\sqrt{155}}$$

Example: 22 The unit vector perpendicular to the plane passing through points P($\hat{i}-\hat{j}+2\hat{k}$), Q($2\hat{i}-\hat{k}$) and R($2\hat{j}+\hat{k}$) is-

(1) $2\hat{i}+\hat{j}+\hat{k}$

(2) $\sqrt{6}\,(2\hat{i}+\hat{j}+\hat{k})$

(3) $\dfrac{1}{\sqrt{6}}\,(2\hat{i}+\hat{j}+\hat{k})$

(4) $\dfrac{1}{6}\,(2\hat{i}+\hat{j}+\hat{k})$

Solution: $\overrightarrow{PQ} = (2\hat{i} - \hat{k}) - (\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} + \hat{j} - 3\hat{k}$

$\overrightarrow{PR} = (2\hat{i} + \hat{k}) - (\hat{i} - \hat{j} + 2\hat{k}) = -\hat{i} + 3\hat{j} - \hat{k}$

Now $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$

$= 8\hat{i} + 4\hat{j} + 4\hat{k}$

$\Rightarrow \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{64+16+16} = 4\sqrt{6}$

$\therefore$ reqd. unit vector $= \dfrac{4(2\hat{i} + \hat{j} + \hat{k})}{4\sqrt{6}}$

$= \dfrac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$

Example: 23 If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, then correct statement is-

(1) $\vec{a} \,||\, (\vec{b} - \vec{c})$ (2) $\vec{a} \perp (\vec{b} - \vec{c})$ (3) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ (4) None of these

Solution: $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ ...(1)

Also $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \,||\, (\vec{b} - \vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \,||\, (\vec{b} - \vec{c})$ ...(2)

Observing to (1) and (2) we find that

$\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$

Example: 24 The components of any vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are-

(1) $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}, \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$ (2) $\dfrac{|\vec{a} \times \vec{b}|}{|\vec{b}|}, \dfrac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$ (3) $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}, \dfrac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$ (4) $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}, \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$

Solution: Let θ be the angle between $\vec{a}$ and $\vec{b}$, then component of $\vec{a}$ along $\vec{b}$

$= |\vec{a}| \cos\theta = \dfrac{|\vec{a}|\,\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

and component of $\vec{a}$ perpendicular to $\vec{b}$

$= |\vec{a}| \sin\theta = \dfrac{|\vec{a}||\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$

Example: 25 The area of parallelogram whose diagonals are respectively $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ is-

(1) $5\sqrt{2}$ (2) $5\sqrt{3}$ (3) $2\sqrt{5}$ (4) $3\sqrt{5}$

Solution: Area of parallelogram $= \dfrac{1}{2}|\vec{a} \times \vec{b}|$

where $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= -2\hat{i}-14\hat{j}-10\hat{k}$

Area of parallelogram

$= \frac{1}{2}\,|-2\hat{i}-14\hat{j}-10\hat{k}\,| = \sqrt{1+49+25} = 5\sqrt{3}$

Example: 26 If $\hat{i}-\hat{j}+2\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$ are position vectors of vertices of a triangle, then its area is-

(1) 26
(2) 13
(3) $2\sqrt{13}$
(4) $\sqrt{13}$

Solution: If A, B, C are given vertices, then

$\overrightarrow{AB} = \hat{i}+2\hat{j}-3\hat{k}$, $\overrightarrow{AC} = 2\hat{i}$

$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = (\hat{i}+2\hat{j}-3\hat{k})\times 2\hat{i} = -4\hat{k}-6\hat{j}$

$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{16+36} = 2\sqrt{13}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{13}$

Example: 27 If A,B,C,D are any four points, then

$|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ equals-

(1) Area of ΔABC
(2) 2(Area of ΔABC)
(3) 3(Area of ΔABC)
(4) 4 (Area of ΔABC)

Solution: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be position vectors of points A,B,C and D respectively, then

$\overrightarrow{AB} \times \overrightarrow{CD} = (\vec{b}-\vec{a})\times(\vec{d}-\vec{c})$

$= \vec{b}\times\vec{d} - \vec{b}\times\vec{c} - \vec{a}\times\vec{d} + \vec{a}\times\vec{c}$

Similarly

$\overrightarrow{BC}\times\overrightarrow{AD} = \vec{c}\times\vec{d} - \vec{c}\times\vec{a} - \vec{b}\times\vec{d} + \vec{b}\times\vec{a}$

$\overrightarrow{CA}\times\overrightarrow{BD} = \vec{a}\times\vec{d} - \vec{a}\times\vec{b} - \vec{c}\times\vec{d} + \vec{c}\times\vec{b}$

Therefore given expression

$= |2\,(\vec{b}\times\vec{a} - \vec{b}\times\vec{c} + \vec{a}\times\vec{c})|$

$= 2\,|(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})| = 4$ (Area of ΔABC)

Example: 28 $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the position vectors of four coplanar points A, B, C and D respectively. If $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0 = (\vec{b}-\vec{d}).(\vec{c}-\vec{a})$, then for the ΔABC, D is-

(1) incentre
(2) orthocentre
(3) circumcentre
(4) centroid

Solution: $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0$

$\Rightarrow (\vec{a}-\vec{d}) \perp (\vec{b}-\vec{c}) \Rightarrow \overrightarrow{AD} \perp \overrightarrow{BC}$

Similarly $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0$

$\Rightarrow \overrightarrow{BD} \perp \overrightarrow{AC}$

$\therefore$ D is the orthocentre of ΔABC.

Example:29 Force $\hat{i}+2\hat{j}-3\hat{k}$, $2\hat{i}+3\hat{j}+4\hat{k}$ and $-\hat{i}-\hat{j}+\hat{k}$ are acting at the point P (0, 1, 2). The moment of these forces about the point A (1, –2, 0) is-

(1) $2\hat{i}-6\hat{j}+10\hat{k}$
(2) $-2\hat{i}+6\hat{j}-10\hat{k}$
(3) $2\hat{i}+6\hat{j}-10\hat{k}$
(4) None of these

Solution: If $\vec{F}$ be the resultant force, then

$\vec{F} = 2\hat{i}+4\hat{j}+2\hat{k}$

Also $\vec{r} = \overrightarrow{AP} = -\hat{i}+3\hat{j}+2\hat{k}$

$\therefore$ required moment = $\vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 2 \\ 2 & 4 & 2 \end{vmatrix} = -2\hat{i}+6\hat{j}-10\hat{k}$$

Example: 30 $\vec{a}.(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})$ is equal to-

(1) 0 (2) 2 [abc] (3) [abc] (4) None of these

Solution: $\vec{a}.(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})$

$= \vec{a}.[(\vec{b}+\vec{c})\times\vec{a}+(\vec{b}+\vec{c})\times\vec{b}+(\vec{b}+\vec{c})\times\vec{c}]$

$= \vec{a}.[(\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{b}\times\vec{b}+\vec{c}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{c}]$

$= \vec{a}.[\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{c}\,\vec{a}]$

$= 0$

Example: 31 If vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}+p\hat{j}+5\hat{k}$ are coplanar, then the value of p is-

(1) 2 (2) 6 (3) – 2 (4) – 6

Solution: $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & p & 5 \end{vmatrix} = 0$$

$\Rightarrow (10 + p + 3) - (6 - 5 - p) = 0$

$\Rightarrow\ p = -6$

Example:32 If $\vec{a}=-3\hat{i}+8\hat{j}+5\hat{k}$, $\vec{b}=-3\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$ are the coterminous edges of a parallelopiped then its volume is-

(1) 108 (2) 210 (3) 272 (4) 302

Solution: Required volume = $[\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} -3 & 8 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$= |-3(-21-15) - 8(9+21) + 5(15-49)|$

$= |108 - 240 - 170|$

$= 302$

Example:33 For any vector $\vec{a}$, $\vec{u} = \hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k})$ equals-

(1) $2\vec{a}$ (2) $-2\vec{a}$ (3) $\vec{a}$ (4) $-\vec{a}$

Solution: $\vec{u} = (\hat{i}.\hat{i})\vec{a} - (\hat{i}.\vec{a})\hat{i} + (\hat{j}.\hat{j})\vec{a} - (\hat{j}.\vec{a})\hat{j}$

$+ (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= \vec{a} - a_1\hat{i} + \vec{a} - a_2\hat{j} + \vec{a} - a_3\hat{k}$

[$\vec{a} = a_1 i + a_2 \hat{j} + a_3 \hat{k}$ (say)]

$\therefore$ u = $3\vec{a} - \vec{a} = 2\vec{a}$

Example:34 Let $\vec{a}, \vec{b}, \vec{c}$ such that $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{c}| = 2$ and if $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = \vec{0}$, then acute angle between $\vec{a}$ and $\vec{c}$ is -

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) None of these

Solution: If angle between $\vec{a}$ and $\vec{c}$ is θ then –

$\vec{a} . \vec{c} = |\vec{a}|\,|\vec{c}| \cos\theta = 1.2 \cos\theta = 2\cos\theta$

but $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = \vec{0}$

$\Rightarrow (\vec{a}.\vec{c})\vec{a} - (\vec{a}.\vec{a})\vec{c} + \vec{b} = \vec{0}$

$\Rightarrow (2\cos\theta)\vec{a} - 1.\ \vec{c} = -\vec{b}$

$\Rightarrow [(2\cos\theta)\vec{a} - \vec{c}]^2 = [-\vec{b}]^2$

$\Rightarrow 4\cos^2\theta\, |\vec{a}|^2 - 2.\,(2\cos\theta)\vec{a}.\vec{c} + |\vec{c}|^2$

$= |\vec{b}|^2$

$\Rightarrow 4\cos^2\theta - 4\cos\theta\,(2\cos\theta) + 4 = 1$

$\Rightarrow 4(1 - \cos^2\theta) = 1 [\because |\vec{a}| = 1, |\vec{b}| = 1]$

$\Rightarrow \sin\theta = 1/2$

$\Rightarrow \theta = \frac{\pi}{6}$

Example:35 Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ & $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of b and c whose projection on 'a' is $\sqrt{\frac{2}{3}}$ will be-

(1) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (2) $2\hat{i} + 3\hat{j} + 3\hat{k}$ (3) $-2\hat{i} - \hat{j} + 5\hat{k}$ (4) $2\hat{i} + \hat{j} + 5\hat{k}$

Solution: Let the required vector $\vec{r} = \vec{b} + t\vec{c}$

$\Rightarrow \vec{r} = (1 + t)\hat{i} + (2 + t)\hat{j} - (1 + 2t)\hat{k}$

Also projection of $\vec{r}$ on $\vec{a} = \sqrt{\frac{2}{3}}$

$\Rightarrow \frac{\vec{r}.\vec{a}}{|\vec{a}|} = \sqrt{2/3}$

$= \frac{2(1+t) - (2+t) - (1+2t)}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

$\Rightarrow -t - 1 = 2$

$\Rightarrow t = -3$

$\therefore \vec{r} = -2\hat{i} - \hat{j} + 5\hat{k}$

Example : 36 If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$ = 0 and $\vec{A}$ = (1,a,a^2),

$\vec{B}$ = (1, b, b^2) and $\vec{C}$ = (1, c, c^2) are non- coplanar vectors, then (abc) equals-

(1) 0 (2) 1 (3) – 1 (4) 2

Solution: Since $\vec{A}$, $\vec{B}$, $\vec{C}$ are non-coplanar vector

$$\therefore \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

Now $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$ = 0

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow (abc+1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow$ abc + 1 = 0 $[\because \Delta \neq 0]$

$\therefore$ abc = – 1

Example:37 A unit vector in xy – plane which makes 45° angle with vector $\hat{i}+\hat{j}$ and 60° angle with vector 3i – 4j will be-

(1) $\hat{i}$ (2) $\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$ (3) $\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$ (4) None of these

Solution: Let the required vector be $\vec{r} = x\hat{i}+y\hat{j}$

$\therefore x^2 + y^2 = 1$ ...(1)

If given vectors be a and b respectively, then as given

$\cos 45° = \frac{\vec{r}.\vec{a}}{|\vec{r}||\vec{a}|} \Rightarrow x + y = 1$ ...(2)

$\cos 60° = \frac{\vec{r}.\vec{b}}{|\vec{r}||\vec{b}|} \Rightarrow 6x - 8y = 5$ ...(3)

But (1), (2) and (3) do not hold together. Hence such a vector is not possible.

Example : 38 If vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ ($a \neq b \neq c \neq 1$) are coplanar, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ equals-

(1) 1 (2) 0 (3) – 1 (4) None of these

Solution: Since vectors are coplanar,

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 0 & 1-b & c-1 \end{vmatrix} = 0$$

[Using $R_2 - R_1$, $R_3 - R_2$]

$\Rightarrow$ a(b – 1) (c – 1) – (1 – a)[(c – 1) – (1 – b)] = 0

$\Rightarrow$ a (1 – b) (1 – c) + (1 – a) (1 – c) + (1– a) (1– b) = 0

$\Rightarrow$ (a – 1 + 1) (1 – b) (1 – c) + (1 – a) (1 – c) + (1 – a) (1 – b) = 0

$\Rightarrow$ (1 – b) (1 – c) + (1 – a) (1 – c) + (1 – a) (1 – b) = (1 – a) (1 – b) (1 – c)

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Example : 39 Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$ if vector $\vec{c}$ is such that $\vec{a} . \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$ is the 30° then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to -

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) 2 (4) 3

Solution: $|\vec{c} - \vec{a}|^2 = (\vec{c} - \vec{a}) . (\vec{c} - \vec{a}) = (2\sqrt{2})^2$

$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c} . \vec{a} = 8$

$\Rightarrow |\vec{c}|^2 + (4 + 1 + 4) - 2\vec{c} . \vec{a} = 8$

$\Rightarrow |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$ $[\because \vec{a} . \vec{c} = |\vec{c}|]$

$\Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0$

$|\vec{c} - 1|^2 = 0 \Rightarrow \vec{c} = 1$

$|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}|\,|\vec{c}| \sin 30^\circ$

$= 1 \times \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\vec{a} \times \vec{b}|$

$$\text{But} \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{4 + 4 + 1} = 3$

$\therefore |(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{3}{2}$