KGS IAS

Maths Optional

By Dhrub Singh Sir

Proof: Let $\langle a_n \rangle$ is a monotonically increasing seq. which is not bdd above.

Let $G > 0$ be any number (May be very large). $\because \langle a_n \rangle$ is not bdd above. $\therefore \exists$ a term a_m s.t.

$$a_m > G.$$

Cor 1: A monotonically increasing seq. which is bdd above must converge to its supremum (l.u.b.). And a monotonically decreasing seq. which is bdd below must converge to its infimum (g.l.b.).

Cor 2: Every monotonically increasing seq. which is not bdd above, diverges to $+\infty$.

<u>Cor. 3</u>: Every monotonically decreasing seq. which is not <u>bdd</u> <u>below</u>, diverges to $-\infty$.

<u>Prob</u>: Show that the seq. $\langle s_n \rangle$, where

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

is cgt.

$\because$ seq. $\langle a_n \rangle$ is monotonically increasing.

$\therefore$ $a_n \geq a_m > G \quad \forall n \geq m$

$\therefore$ $a_n > G \quad \forall n \geq m$

$\therefore$ $\langle a_n \rangle \longrightarrow +\infty$.

$$s_{n+1}-s_n=\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}$$

$$=\frac{2n+2+2n+1-2(2n+1)}{(2n+1)\cdot 2\cdot(n+1)}$$

$$=\frac{\cancel{4}n+3-\cancel{4}n-2}{(2n+1)\cdot 2\cdot(n+1)}>0$$

$\therefore s_{n+1}-s_n>0$

$\Rightarrow s_{n+1}>s_n \quad \forall n\in\mathbb{N}.$

$\therefore \langle s_n\rangle$ — mono. $\uparrow$.

$$\rightarrow s_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$$

$$s_{n+1}=\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+2}$$

$$s_{n+1}-s_n=\cancel{\frac{1}{n+2}}+\cancel{\frac{1}{n+3}}+\cdots$$

$$+\cancel{\frac{1}{2n}}+\frac{1}{2n+1}+\frac{1}{2n+2}-\left(\frac{1}{n+1}+\cancel{\frac{1}{n+2}}\right.$$

$$\left.+\cdots+\cancel{\frac{1}{2n}}\right)$$

Prob: Show that the sequence $\langle s_n \rangle$, where

$$s_n = \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n}$$

is cgt.

$\rightarrow$ $s_{n+1} - s_n = \frac{1}{\lfloor n+1} > 0 \quad \forall n$.

$\therefore s_{n+1} > s_n \quad \forall n \in \mathbb{N}$.

$\therefore \langle s_n \rangle$ is mono. $\uparrow$.

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + \cdots \text{ n times}$$

$$= n \cdot \frac{1}{n} = 1$$

$\therefore s_n < 1 \quad \forall n \in \mathbb{N}$.

$\therefore \langle s_n \rangle$ is bdd above.

$\because \langle s_n \rangle$ is monotonically increasing and bdd above.

$\therefore$ St is cgt.

$\therefore s_n < 2 \quad \forall n \in \mathbb{N}$.

$\langle s_n \rangle$ is mono. $\uparrow$ and is bdd. above.

$\therefore$ ST is cgt.

$$s_n = \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{1}{\underline{|n}}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}} < 2$$

$\left[\because \underline{|n} > 2^{n-1} \quad \forall n > 2\right]$

$$\Rightarrow \sqrt{3\cdot s_{k+1}} > \sqrt{3\cdot s_k}$$

$$\Rightarrow s_{k+2} > s_{k+1}$$

$\therefore$ By PMI,

$$s_{n+1} > s_n \quad \forall n \in \mathbb{N}$$

$\therefore$ $\langle s_n \rangle$ is mono. $\uparrow$

$$s_1 = 1 < 3$$

$$s_2 = \sqrt{3} < 3$$

Prob: Show that the seq. $\langle s_n \rangle$ defined by the recursion formula $s_{n+1} = \sqrt{3\cdot s_n}$, $s_1 = 1$ converges to 3.

$\rightarrow$ $s_1 = 1$, $s_2 = \sqrt{3}$, $s_3 = \sqrt{3\sqrt{3}}$

$$s_2 > s_1, \quad s_3 > s_2$$

Suppose $s_{k+1} > s_k$

$$\Rightarrow 3\cdot s_{k+1} > 3\cdot s_k$$

$\because$ $\langle s_n \rangle$ is mono. $\uparrow$ and is bdd above.

$\therefore$ it is cgt.

Let $\lim s_n = l$.

$\because$ $s_{n+1} = \sqrt{3 s_n}$

$\Rightarrow s_{n+1}^2 = 3 s_n$

$\Rightarrow \lim s_{n+1}^2 = \lim 3 \cdot s_n$

$\Rightarrow l^2 = 3l$

$\Rightarrow l^2 - 3l = 0$

Suppose $s_K < 3$

$\Rightarrow 3 \cdot s_K < 3 \cdot 3$

$\Rightarrow \sqrt{3 \cdot s_K} < \sqrt{9}$

$\Rightarrow s_{K+1} < 3$

$\therefore$ By PMI,

$s_n < 3 \quad \forall n \in \mathbb{N}$.

$\therefore$ $\langle s_n \rangle$ is bdd. above.

Prob: Let $\langle a_n \rangle$ be a seq. defined by $a_1 = 1$, $a_{n+1} = \frac{3+2a_n}{2+a_n}$, $n \geq 1$.

Show that the seq. $\langle a_n \rangle$ is cgt and find its limit.

$\rightarrow \langle 1, \frac{5}{3}, \frac{19}{11}, \quad \rangle$

$a_2 > a_1$.

$a_3 > a_2$

Suppose $a_{k+1} > a_k$

$\Rightarrow a_{k+1} - a_k > 0$

$l(l-3) = 0$

$\Rightarrow l = 0, 3$

$l \neq 0$ As $s_n \geq 1$ $\forall n$.

$\therefore l = 3$

$\Rightarrow a_{k+2} - a_{k+1} > 0$

$\Rightarrow a_{k+2} > a_{k+1}$

$\therefore$ By PMI, $a_n > a_{n+1}$ $\forall n$

$\langle a_n \rangle$ is mono. $\uparrow$.

$$a_{n+1} = \frac{3+2a_n}{2+a_n}$$

$$= \frac{4+2a_n-1}{2+a_n}$$

$$= 2 - \frac{1}{2+a_n} < 2$$

$$a_{k+2} - a_{k+1} = \frac{3+2a_{k+1}}{2+a_{k+1}} - \frac{3+2a_k}{2+a_k}$$

$$= \frac{\cancel{6} + 4a_{k+1} + 3a_k + \cancel{2a_k a_{k+1}} - \cancel{6} - 4a_k - 3a_{k+1} - \cancel{2a_k a_{k+1}}}{(2+a_{k+1})(2+a_k)}$$

$$= \frac{a_{k+1} - a_k}{(2+a_{k+1})(2+a_k)} > 0$$

As $a_{k+1} > a_k$.

$$l = \frac{3+2l}{2+l}$$

$$\Rightarrow \cancel{2l} + l^2 = 3 + \cancel{2l}$$

$$\Rightarrow l^2 = 3$$

$$\Rightarrow l = \pm\sqrt{3}$$

$\boxed{l \neq -\sqrt{3} \quad \text{as } a_n \geq 1 \ \forall n}$

$\therefore \underline{l = \sqrt{3}}$

$\therefore a_{n+1} < 2 \quad \forall n \in \mathbb{N}$

$\therefore \langle a_n \rangle$ is mono. $\uparrow$ and is bdd. above.

$\therefore$ It is cgt.

Let $\lim a_n = l$.

$$a_{n+1} = \frac{3+2a_n}{2+a_n}$$

$$\Rightarrow \lim a_{n+1} = \lim \frac{3+2a_n}{2+a_n}$$

$\Rightarrow 7 + s_{k+1} > 7 + s_k$

$\Rightarrow \sqrt{7+s_{k+1}} > \sqrt{7+s_k}$

$\Rightarrow s_{k+2} > s_{k+1}$

$\therefore$ By PMI.

$\langle s_n \rangle$ is mono. $\uparrow$.

$s_1 = \sqrt{7} < 7$

Suppose $s_k < 7$

$\Rightarrow 7 + s_k < 7 + 7$

<u>prob:</u> prove that the seq. $\langle s_n \rangle$ defined by

$$s_{n+1} = \sqrt{7+s_n}, \; s_1 = \sqrt{7}$$

Converge to the +ve root of $x^2 - x - 7 = 0$.

$\rightarrow s_1 = \sqrt{7}, \; s_2 = \sqrt{7+\sqrt{7}}, \ldots$

$s_2 > s_1$

Suppose

$s_{k+1} > s_k$

Let $\lim s_n = l$.

$\because s_{n+1} = \sqrt{7+s_n}$

$\Rightarrow s_{n+1}^2 = 7 + s_n$

$\Rightarrow \lim(s_{n+1}^2) = \lim(7+s_n)$

$\Rightarrow l^2 = 7 + l$

$\Rightarrow l^2 - l - 7 = 0$

$\therefore$ l is the +ve root of

$x^2 - x - 7 = 0$ $[\because s_n \geq \sqrt{7}]$

$\Rightarrow \sqrt{7+s_k} < \sqrt{7+7} < 7$

$\Rightarrow s_{k+1} < 7$

$\therefore$ By PMI,

$s_n < 7 \quad \forall n \in \mathbb{N}$.

$\therefore \langle s \rangle$ is bdd above.

$\because \langle s_n \rangle$ is mono. $\uparrow$ and is bdd above.

$\therefore$ it is cgt.

$\Rightarrow s_{k+1}^2 > s_k^2$

$\Rightarrow \dfrac{3+s_{k+1}^2}{2} > \dfrac{3+s_k^2}{2}$

$\Rightarrow \left(\dfrac{3+s_{k+1}^2}{2}\right)^{1/2} > \left(\dfrac{3+s_k^2}{2}\right)^{1/2}$

$\Rightarrow s_{k+2} > s_{k+1}$.

$\therefore$ By PMI, $s_{n+1} > s_n$ $\forall n$.

$\therefore \langle s_n \rangle$ is mono. $\uparrow$.

<u>prob</u>: A seq. $\langle s_n \rangle$ is defined by $s_1 = 1$, $s_{n+1} = \left(\dfrac{3+s_n^2}{2}\right)^{1/2}$.

Show that $\langle a_n \rangle \to \sqrt{3}$

$\rightarrow$ $s_1 = 1$, $s_2 = \left(\dfrac{3+1}{2}\right)^{1/2}$

$= \sqrt{2}$

$s_2 > s_1$

Suppose

$s_{k+1} > s_k$

$\Rightarrow \left(\frac{3+s_k^2}{2}\right)^{\frac{1}{2}} \le \sqrt{3}$

$\Rightarrow \quad s_{k+1} \le \sqrt{3}$

$\therefore$ By PMI,

$s_n \le \sqrt{3}$ $\forall n$.

$\therefore \langle s_n \rangle$ is bdd above.

$\therefore$ It is cgt.

Let $\lim s_n = l$.

$s_1 = 1 < \sqrt{3}$

$s_2 = \sqrt{2} < \sqrt{3}$

Suppose

$s_k \le \sqrt{3}$

$\Rightarrow \quad s_k^2 \le 3$

$\Rightarrow \quad \frac{3+s_k^2}{2} \le \frac{3+3}{2}$

$l = \pm\sqrt{3}.$

$\boxed{l \neq -\sqrt{3} \quad \text{as } s_n \geq 1}$

$\therefore\ l = \sqrt{3}$

HW

Prob: Let $\langle s_n \rangle$ be defined by $s_1 = 1$, $s_{n+1} = \frac{s_n + 1}{3}$ $\forall n \geq 1$.

Show that $\langle s_n \rangle$ converges to $\frac{1}{2}$.

Given

$$s_{n+1} = \left(\frac{3 + s_n^2}{2}\right)^{1/2}$$

$$\Rightarrow s_{n+1}^2 = \frac{3 + s_n^2}{2}$$

$$\Rightarrow \lim s_{n+1}^2 = \lim \frac{3 + s_n^2}{2}$$

$$\Rightarrow l^2 = \frac{3 + l^2}{2}$$

$$\Rightarrow 2l^2 = 3 + l^2$$

$$\Rightarrow l^2 = 3$$

$\Rightarrow \; -\sqrt{1-a_K} < 0$

$\Rightarrow \; 1-\sqrt{1-a_K} < 1$

$\Rightarrow \; a_{K+1} < 1$

$\therefore$ By PMI

$a_n < 1 \quad \forall n.$

obviously

$0 < a_n$

$\therefore \; 0 < a_n < 1 \quad \forall n$

<u>prob</u>: Show that the seq. $\langle a_n \rangle$ defined by

$$a_{n+1} = 1-\sqrt{1-a_n} \quad \forall n \geq 1$$

and $a_1 < 1$, converges to 0.

$\rightarrow \quad a_2 = 1-\sqrt{1-a_1} < 1$

Suppose $a_K < 1$

$\Rightarrow \; 1-a_K > 0$

$\Rightarrow \; \sqrt{1-a_K} > 0$

Reason: $a_n < 1$

$\Rightarrow 1 - a_n > 0$

$\Rightarrow \sqrt{1-a_n} > 0$

$\Rightarrow 1 + \sqrt{1-a_n} > 1$

$\Rightarrow \dfrac{1}{1+\sqrt{1-a_n}} < 1$

$\therefore a_{n+1} < a_n \quad \forall n \in \mathbb{N}$

$$a_{n+1} = 1 - \sqrt{1-a_n}$$

$$= \frac{\left(1 - \sqrt{1-a_n}\right)\left(1 + \sqrt{1-a_n}\right)}{1+\sqrt{1-a_n}}$$

$$= \frac{\cancel{1} - (\cancel{1} - a_n)}{1+\sqrt{1-a_n}}$$

$$= \frac{a_n}{1+\sqrt{1-a_n}} < a_n$$

$$a_{n+1}^2 - 2a_{n+1} + \cancel{1} = \cancel{1} - a_n$$

$$\Rightarrow a_{n+1}^2 - 2a_{n+1} + a_n = 0$$

$$\lim \left(a_{n+1}^2 - 2a_{n+1} + a_n\right) = 0$$

$$\Rightarrow l^2 - 2l + l = 0$$

$$\Rightarrow l^2 - l = 0$$

$$\Rightarrow l(l-1) = 0$$

$$\Rightarrow l = 0, \cancel{1}$$

$\therefore \langle a_n \rangle$ is mon. $\downarrow$,

<u>Also</u> it is bounded below

$\therefore$ it is cgt

Let $\lim a_n = l$

$$a_{n+1} = 1 - \sqrt{1 - a_n}$$

$$\Rightarrow a_{n+1} - 1 = -\sqrt{1 - a_n}$$

$$\Rightarrow (a_{n+1} - 1)^2 = 1 - a_n$$