KGS IAS

Maths Optional

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Prob: Show that

$$\lim_{n\to\infty}\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+n)^2}\right]=0$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\cdots+\frac{n}{(n+n)^2}\right]=0$$

Let $a_n=\frac{n}{(n+n)^2}$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{4n}=0$$

Prob: Show that

$$\lim_{n\to\infty}\frac{1}{n}\left[1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+4^{\frac{1}{4}}+\cdots+n^{\frac{1}{n}}\right]=1.$$

Soln: $\lim_{n\to\infty} n^{\frac{1}{n}}=1$.

Let $a_n=n^{\frac{1}{n}}$.

By Cauchy's 1st theorem on limits,

$$\lim_{n\to\infty}\frac{1+2^{\frac{1}{2}}+\cdots+n^{\frac{1}{n}}}{n}=1.$$

Cauchy's 2nd th. on limits

If $\lim a_n = l$ where $a_n > 0$ $\forall n \in \mathbb{N}$, then $\lim (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = l$.

$\rightarrow \because \lim a_n = l$.

$\therefore \lim \log(a_n)$

$= \log(\lim a_n)$

$= \log l$.

By Cauchy's 1st th. on limits.

prob: Show that

$$\lim\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$$

$$\rightarrow \lim \underbrace{\frac{1}{n^2}}_{0} + \lim \underbrace{\left[\frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+n)^2}\right]}_{0}$$

Cor: suppose $\langle a_n \rangle$ is a seq. of +ve real numbers s.t. $\lim \left(\frac{a_{n+1}}{a_n}\right) = a, (a>0)$.

Then $\lim (a_n)^{\frac{1}{n}} = a$.

proof: Let $a_0 = 1$

Also $\lim \frac{a_n}{a_{n-1}} = a$

By the main th,

$$\lim \frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n} = \log l.$$

$$\Rightarrow \lim \frac{1}{n} \log (a_1 a_2 \cdots a_n) = \log l.$$

$$\Rightarrow \lim \log (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = \log l.$$

$$\Rightarrow \log \left[\lim (a_1 a_2 \cdots a_n)^{\frac{1}{n}}\right] = \log l$$

$$\Rightarrow \boxed{\lim (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = l}$$

Prob: Prove that if a sequence $\langle a_n \rangle$ of positive numbers converge to a limit l, then

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = l.$$

Deduce that $\lim_{n\to\infty} (n)^{1/n} = 1$

$\rightarrow \langle a_n \rangle, \quad a_1 = 1$

Define, $a_n = \dfrac{n}{n-1} \quad \forall\, n \geq 2$

$$\lim \left(\frac{a_1}{a_0} \times \frac{\cancel{a_2}}{\cancel{a_1}} \times \cancel{\cdots \times} \frac{a_n}{\cancel{a_{n-1}}} \right)^{1/n} = a$$

i.e. $\lim \left(\dfrac{a_n}{a_0} \right)^{1/n} = a$

i.e. $\lim (a_n)^{1/n} = a.$

i.e. $\lim (n)^{1/n} = 1$.

Prob: If $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, then show that

$$\lim_{n\to\infty} \left[\frac{2}{1} \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n \right]^{\frac{1}{n}} = e.$$

$\rightarrow$ Let $a_n = \left(1+\frac{1}{n}\right)^n$

$$= \left(\frac{n+1}{n}\right)^n$$

$$\lim a_n = \lim \frac{n}{n-1}$$

$$= \lim_{n\to\infty} \frac{1}{1-1/n}$$

$$= \frac{1}{1-0} = 1.$$

By the main th,

$$\lim (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = 1$$

i.e. $\lim \left(1 \cdot \cancel{\frac{2}{1}} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{4}{3}} \cdots \cancel{\frac{n}{n-1}}\right)^{\frac{1}{n}} = 1$

$\rightarrow$ Let $a_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$.

$\underline{\lim a_n = e}$.

$\therefore \lim \left[a_1 \cdot a_2 \cdots a_n\right]^{\frac{1}{n}} = e$

$\Rightarrow \lim \left[\frac{2}{1} \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e$

$\Rightarrow \lim \left[\frac{(n+1)^n}{1 \times 2 \times \cdots \times n}\right]^{\frac{1}{n}} = e$

$\lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$.

$\therefore \lim \left[\frac{2}{1} \times \left(\frac{3}{2}\right)^2 \times \left(\frac{4}{3}\right)^3 \times \cdots \times \left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e$

$\underline{\text{Proof:}}$ $\lim\limits_{n\to\infty} \left(\frac{n^n}{\lfloor n}\right)^{\frac{1}{n}}$ exists and equals e.

$$\Rightarrow \lim \left(1+\frac{1}{n}\right) \times \lim \left[\frac{n^n}{\lfloor n}\right]^{\frac{1}{n}} = e$$

$$\Rightarrow \boxed{\lim \left(\frac{n^n}{\lfloor n}\right)^{\frac{1}{n}} = e}$$

<u>Prob</u>: prove that $\lim\limits_{n\to\infty} \left\{\frac{\lfloor 3n}{(\lfloor n)^3}\right\}^{\frac{1}{n}}$

$= 27.$

$$\lim \left[\frac{(n+1)^n}{\lfloor n}\right]^{\frac{1}{n}} = e$$

$$\Rightarrow \lim (n+1) \left[\frac{1}{\lfloor n}\right]^{\frac{1}{n}} = e$$

$$\Rightarrow \lim (n+1) \left[\frac{n^n}{\lfloor n \cdot n^n}\right]^{\frac{1}{n}} = e$$

$$\Rightarrow \lim \left(\frac{n+1}{n}\right) \times \left[\frac{n^n}{\lfloor n}\right]^{\frac{1}{n}} = e$$

$$\text{Let } a_n = \frac{\underline{|3n}}{\left(\underline{|n}\right)^3}$$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\underline{|3n+3}}{\left(\underline{|n+1}\right)^3}}{\dfrac{\underline{|3n}}{\left(\underline{|n}\right)^3}}$$

$$= \frac{\underline{|3n+3}}{\left(\underline{|n+1}\right)^3} \times \frac{\left(\underline{|n}\right)^3}{\underline{|3n}}$$

$$= \frac{(3n+3)(3n+2)\times(3n+1)\times\cancel{\underline{|3n}}\ \cancel{\times\left(\underline{|n}\right)^3}}{\cancel{\underline{|3n}} \times \left((n+1)\cdot\cancel{\underline{|n}}\right)^3}$$

$$= \frac{3\cancel{(n+1)}(3n+2)(3n+1)}{(n+1)^{\cancel{3}\ 2}}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3\times\cancel{n}\cdot\left(3+\frac{2}{n}\right)\cdot\cancel{n}\cdot\left(3+\frac{1}{n}\right)}{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}$$

$$= \frac{3\times3\times3}{1} = 27.$$

Let $b_n = \dfrac{(n+1)(n+2)\times \cdots (n+n)}{n^n}$

$$\frac{b_{n+1}}{b_n} = \frac{\dfrac{(n+2)(n+3)\cdots(2n+2)}{(n+1)^{n+1}}}{\dfrac{(n+1)(n+2)\cdots(2n)}{n^n}}$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^{n+1}} \times \frac{n^n}{n+1}$$

$\therefore \lim (a_n)^{1/n} = 27$

Prob: prove that if

$$a_n = \frac{1}{n}\left\{(n+1)(n+2)\times \cdots (n+n)\right\}^{\frac{1}{n}}$$

then $\langle a_n \rangle \to \dfrac{4}{e}$.

$\rightarrow a_n = \left\{\dfrac{(n+1)(n+2)\cdots(n+n)}{n^n}\right\}^{\frac{1}{n}}$

$\therefore \lim (b_n)^{1/n} = 4/e$

$\therefore \lim a_n = 4/e$

$$= \left(\frac{n}{n+1}\right)^n \cdot \frac{(2n+1)\cdot 2\cancel{(n+1)}}{(n+1)^{\cancel{2}}}$$

$$\lim \frac{b_{n+1}}{b_n} = \lim \cdot \frac{1}{\left(\frac{n+1}{n}\right)^n} \times \frac{\cancel{n}(2+\frac{1}{n})\times 2}{\cancel{n}(1+\frac{1}{n})}$$

$$= \lim \frac{1}{(1+\frac{1}{n})^n} \times \frac{(2+\frac{1}{n})\cdot 2}{(1+\frac{1}{n})}$$

$$= \frac{1}{e} \times 2\cdot 2 = \frac{4}{e}.$$

Th: If $\langle a_n \rangle$ be a sequence such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$, then $\lim a_n = 0$

Proof: Choose $\varepsilon > 0$ s.t.

$$|l| + \varepsilon < 1$$

Let $K = |l| + \varepsilon$

$$\therefore \underline{K < 1}$$

$\because \lim \frac{a_{n+1}}{a_n} = l$.

$\therefore$ corr. to $\varepsilon > 0$, $\exists$ a +ve int. m s.t.

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \quad \forall n \geq m$$

$$\left|\frac{a_{n+1}}{a_n}\right| - |l| \leq \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \quad \forall n \geq m$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| - |l| < \varepsilon \quad \forall n \geq m$$

$$\Rightarrow \frac{|a_n|}{|a_m|} < K^{n-m} \quad \forall n \geq m$$

$$\Rightarrow |a_n| < \frac{|a_m|}{K^m} \cdot K^n \quad \text{---} (11) \quad \forall n \geq m$$

$\because 0 < K < 1$

$\therefore K^n \to 0$

$\therefore (11) \Rightarrow$

$a_n \to 0$

i.e. $\lim a_n = 0$.

$$\therefore \left|\frac{a_{n+1}}{a_n}\right| < \underbrace{|l| + \varepsilon}, \quad \forall n \geq m$$

$$\therefore \frac{|a_{n+1}|}{|a_n|} < K, \quad \forall n \geq m \quad \text{---} (1)$$

Put $n = m, m+1, m+2, \ldots, n-1$ in (1) successively and then multiplying.

$$\frac{|a_{m+1}|}{|a_m|} \cdot \frac{|a_{m+2}|}{|a_{m+1}|} \times \cdots \frac{|a_n|}{|a_{n-1}|} < K \cdot K \cdot \ldots \; (n-m) \text{ times.}$$

s.t

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \ \forall n \geq m$$

$$\Rightarrow l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon, \ \forall n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > l - \varepsilon \quad \forall n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > K \quad \forall n \geq m \quad \text{(1)}$$

Th: If $\langle a_n \rangle$ be a sequence such that $\lim \frac{a_{n+1}}{a_n} = l > 1$ then $\lim a_n = \infty$.

proof: choose $\varepsilon > 0$ s.t

$l - \varepsilon > 1$, let $K = l - \varepsilon$

$\therefore K > 1$

$\because \lim \frac{a_{n+1}}{a_n} = l$.

$\therefore$ corr. to $\varepsilon > 0$, $\exists$ a +ve int m

$\because K > 1$

$\therefore K^n \to \infty$

$\therefore$ from (11)

$a_n \to \infty$

i.e. $\lim a_n = \infty$.

Put $n = m, m+1, m+2, \cdots, n-1$

in successively and then multiplying.

$$\frac{a_{m+1}}{a_m} \times \frac{a_{m+2}}{a_{m+1}} \times \cdots \times \frac{a_n}{a_{n-1}} > K^{n-m}$$

$\forall n \geq m$

$$\Rightarrow \frac{a_n}{a_m} > \frac{K^n}{K^m} \quad \forall n \geq m$$

$$\Rightarrow a_n > \frac{a_m}{K^m} \cdot K^n, \quad \forall n \geq m \quad \text{—— (11)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{\cancel{2^{n+1}}_{2}} \times \frac{\cancel{2^n}}{n^2}$$

$$= \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \times \frac{1}{2}$$

$$= 1 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

$\therefore \underline{\lim a_n = 0}$.

Prob: Show that

$$\lim_{n\to\infty} 2^{-n} \cdot n^2 = 0.$$

$\longrightarrow$ Let $a_n = 2^{-n} \cdot n^2$

$$= \frac{n^2}{2^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2}{2^{n+1}}}{\dfrac{n^2}{2^n}}$$