KGS IAS

Maths Optional

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$$\Rightarrow dy = \frac{2p^2 - 2 + 2}{1-p}\,dp$$

$$\text{i.e. } dy = \left[\frac{2(p^2-1)}{1-p} + \frac{2}{1-p}\right]dp$$

$$dy = \left[\frac{-2\cancel{(p-1)}(p+1)}{\cancel{1-p}} + \frac{2}{1-p}\right]dp$$

$$dy = -\left[\frac{2}{p-1} + 2(p+1)\right]dp$$

$$dy = -2\left[\frac{1}{p-1} + (p+1)\right]dp$$

<u>prob:</u> $x = y + p^2$ —— (1)

$\rightarrow$ Diff. w.r.t 'y'

$$\frac{1}{p} = 1 + 2p \cdot \frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} - 1 = 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1-p}{p} = 2p\frac{dp}{dy}$$

$$\Rightarrow dy = \frac{2p^2}{1-p}\,dp$$

(II) and (III) constitute the sol^n. p is a parameter.

Integ.

$$y = -2\left[\log(p-1) + \frac{p^2}{2} + p\right] + C$$

$$\Rightarrow y = c - \left[2p + p^2 + 2\log(p-1)\right] \quad \text{(II)}$$

using this in (I)

$$x = c - \left[+2p + \cancel{p^2} + 2\log(p-1)\right]$$

i.e. $x = c - \left[2p + 2\log(p-1)\right] + \cancel{p^2}$ (III)

$$dy = \left[-\frac{3a}{p^3} - \frac{2b}{p^2}\right] dp$$

int.

$$y = \left[\frac{-3}{-2}\cdot\frac{a}{p^2} - \frac{2\ b}{-1\cdot p}\right] + c$$

$$\Rightarrow y = \frac{3a}{2p^2} + \frac{2b}{p} + c \quad \text{—(ii)}$$

(i) and (ii) constitute the sol^n.

where p is a parameter.

Prob: $xp^3 = a + bp$

$$\Rightarrow x = \frac{a}{p^3} + \frac{b}{p^2} \quad \text{—— (i)}$$

Diff. w.r.t. y.

$$\frac{dx}{dy} = \left[\frac{a(-3)}{p^4} + \frac{b(-2)}{p^3}\right]\cdot\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \left[-\frac{3a}{p^4} - \frac{2b}{p^3}\right]\frac{dp}{dy}$$

$$\frac{dp}{dx} = 0$$

$$\Rightarrow dp = 0$$

int

$$p = c$$

Put in ①

$$\boxed{y = cx + f(c)}$$

Clairaut's eqn

$$y = xp + f(p) \longrightarrow ①$$

Diff. w.r.t. 'x',

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\Rightarrow \underbrace{\left[x + f'(p)\right]}_{\text{Singular sol}^n} \frac{dp}{dx} = 0$$

prob: $(y-px)(p-1)=p$

$\rightarrow \quad y-px=\frac{p}{p-1}$

$\Rightarrow \quad y=px+\frac{p}{p-1}$

Clairaut's form.

$\therefore$ solⁿ. is

$$\boxed{y=cx+\frac{c}{c-1}}$$

prob: $y=px+\frac{m}{p}$

Clairaut's form.

$\rightarrow$ solⁿ. is

$$\boxed{y=cx+\frac{m}{c}}$$

$$\frac{dv}{du} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} \cdot \frac{dv}{du}$$

$$p = \frac{x}{y} P \qquad \left(P = \frac{dv}{du}\right)$$

$① \Rightarrow$

$$\frac{x^2}{y\cancel{2}} P^2 \cdot \cancel{xy} + \frac{\cancel{x}}{y} P \cdot x^2 - \frac{\cancel{x}}{\cancel{y}} P \cdot y^{\cancel{2}} - \cancel{x}y = a^2 \cdot \frac{x}{y} P$$

Reduction to Clairaut's form

① using the transformation.

$u = x^2, \quad v = y^2$

Prob: $(px - y)(py + x) = a^2 \cdot p.$

$\longrightarrow p^2 xy + p \cdot x^2 - p y^2 - xy = a^2 p \quad — ①$

Put $u = x^2, \quad v = y^2$

$du = 2x\,dx \quad dv = 2y\,dy$

$$(pu-v)p+1=a^2p$$

$$\Rightarrow pu-v=\frac{a^2p}{p+1}$$

$$\Rightarrow v=pu-\frac{a^2p}{p+1}$$

Clairaut's form

$\therefore$ Solⁿ. is

$$v=cu-\frac{a^2c}{c+1}$$

i.e. $y^2=cx^2-\dfrac{a^2c}{c+1}$.

$$\frac{x^2}{y}p^2+\frac{x^2}{y}p-py-y=\frac{a^2}{y}p$$

$$\Rightarrow \frac{x^2p^2+x^2p-py^2-y^2}{\cancel{y}}=\frac{a^2}{\cancel{y}}p$$

$$\Rightarrow x^2p^2+x^2p-py^2-y^2=a^2p$$

$$\Rightarrow up^2+up-pv-v=a^2p$$

$$\Rightarrow u\underline{(p^2+p)}-v(p+1)=a^2p$$

$$\frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}$$

$$p = \frac{x}{y}P, \quad P = \frac{dv}{du}.$$

$\therefore$ (1) $\Rightarrow$

$$x^2\left(y - \frac{x}{y}P\cdot x\right) = \cancel{y}\cdot\frac{x^2}{y^{\cancel{2}}}P^2$$

$$\Rightarrow x^2\left(\frac{y^2 - x^2P}{\cancel{y}}\right) = \frac{x^2}{\cancel{y}}P^2$$

$$\Rightarrow \cancel{u}(v - uP) = \cancel{u}\,P^2$$

Prob: Reduce the following eqn. in clairaut's form and hence find the soln.

$$x^2(y - px) = y\cdot p^2 \quad \text{—— (1)}$$

$\longrightarrow$ Put $u = x^2$, $V = y^2$

$$du = 2x\,dx, \quad dv = 2y\,dy$$

$$\frac{dv}{du} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx}$$

Prob: Reduce the following eqn. in Clairaut's form and hence find the soln.

$$xy(y-px) = x+py \quad \text{--- (1)}$$

$\rightarrow$ Put $u = x^2$, $v = y^2$

$du = 2x\,dx$, $dv = 2y\,dy$

$$\frac{dv}{du} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx} \Rightarrow \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}$$

i.e. $p = \frac{x}{y}P$

$$v = uP + P^2$$

Clairaut's form.

$\therefore$ soln. is

$$v = uc + c^2$$

i.e. $\boxed{y^2 = c \cdot x^2 + c^2}$

Clairaut's form.

$\therefore$ sol^n. is

$v = uc + 1 + c$

i.e. $\boxed{y^2 = x^2 \cdot c + 1 + c}$

$(1) \Longrightarrow$

$$xy\left(y - \frac{x}{y}p \cdot x\right) = x + \frac{x}{y}p \cdot y$$

$$\Rightarrow xy\left(\frac{y^2 - x^2 p}{\cancel{y}}\right) = \frac{xy + xy \cdot p}{\cancel{y}}$$

$$\Rightarrow \cancel{xy}\left(y^2 - x^2 p\right) = \cancel{xy}(1+p)$$

$$\Rightarrow v - up = 1 + p$$

$$\Rightarrow v = up + 1 + p$$

$$\frac{dy}{dx} = \frac{e^x}{e^y} \frac{dv}{du}.$$

i.e. $p = \frac{e^x}{e^y} P$, $P = \frac{dv}{du}$

① $\Rightarrow$

$$e^{3x}\left[\frac{e^x}{e^y} \cdot P - 1\right] + \frac{e^{3x}}{\cancel{e^{3y}}} P^3 \cdot \cancel{e^{2y}} = 0$$

e^y

$$\Rightarrow e^{3x}\left[\frac{e^x P - e^y}{e^y}\right] + \frac{e^{3x} \cdot P^3}{e^y} = 0$$

(ii) using the transformation

$u = e^x$, $v = e^y$.

<u>Prob:</u> $e^{3x}(p-1) + p^3 e^{2y} = 0$ ①

$\longrightarrow$ Put $u = e^x$, $v = e^y$

$du = e^x \cdot dx$, $dv = e^y \cdot dy$

$$\frac{dv}{du} = \frac{e^y\, dy}{e^x\, dx}$$

(iii) using the transfor.

$u = \frac{1}{x}, \quad v = \frac{1}{y}$

Prob: Reduce the following eqn. in clairaut's form and hence solve.

$$y^2(y - px) = x^4 p^2 \quad \text{--- (1)}$$

$\Rightarrow [e^x p - e^y] + p^3 = 0$

$\Rightarrow up - v + p^3 = 0$

$\Rightarrow v = up + p^3$

clairaut's form.

$\therefore$ Soln. is

$v = cu + c^3$

i.e. $\boxed{e^y = c \cdot e^x + c^3}$

① ⇒

$$\cancel{y^2}\left(y-\frac{y^2}{x\cancel{x}}p\cdot\cancel{x}\right)=\cancel{x}\cdot\frac{y\cancel{y}^2}{\cancel{x}\cancel{y}}p^2$$

$$\Rightarrow \left(\frac{xy-y^2p}{x}\right)=y^2p^2$$

$$\Rightarrow xy-y^2p=x\cdot y^2p^2$$

$$\Rightarrow \frac{1}{y}-\frac{1}{x}p=p^2$$

$$\Rightarrow v-uP=P^2$$

put $u=\frac{1}{x}$, $v=\frac{1}{y}$

$$du=-\frac{1}{x^2}dx,\quad dv=-\frac{1}{y^2}dy$$

$$\frac{dv}{du}=\frac{+\frac{1}{y^2}dy}{+\frac{1}{x^2}dx}=\frac{x^2}{y^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2}{x^2}\cdot\frac{dv}{du}$$

i.e. $p=\frac{y^2}{x^2}P$, $P=\frac{dv}{du}$

(iv) Miscellaneous trans.

prob: Reduce the eqn.

(2023) $x^2p^2 + y(2x+y)p + y^2 = 0$

to Clairaut's form

by the substitution

$u = y$ and $v = xy$. Hence

solve the eqn. and show

that $y + 4x = 0$ is a singular

soln. of the diff. eqn.

$$v = up + p^2$$

Clairaut's form,

Soln.: u

$$v = cu + c^2$$

i.e. $\dfrac{1}{y} = \dfrac{c}{x} + c^2$

$$\frac{1}{y} = \frac{c + c^2 \cdot x}{x}$$

$\Rightarrow x = cy + c^2 \cdot xy$

$$p = \frac{y}{p} + x, \qquad p = \frac{dv}{du}.$$

$$\Rightarrow p - x = \frac{y}{p}$$

$$\Rightarrow p = \frac{y}{p-x}$$

$$(1) \Rightarrow$$

$$x^2 \cdot \frac{y^2}{(p-x)^2} + \cancel{y}(2x+y)\frac{y}{p-x} + y^2 = 0$$

$$\rightarrow$$

$$x^2p^2 + y(2x+y)\cdot p + y^2 = 0 \quad \text{--- } (1)$$

put $u = y$, $v = xy$

$$\Rightarrow du = dy \; , \; dv = d(xy)$$

$$= y\,dx + x\,dy$$

$$\frac{dv}{du} = \frac{y\,dx + x\,dy}{dy}$$

$$= y\frac{dx}{dy} + x$$

$$pu - v + p^2 = 0$$

$$\Rightarrow v = pu + p^2$$

Clairaut's form.

$\therefore$ soln. is

$$v = cu + c^2$$

i.e. $\boxed{xy = cy + c^2}$

$$\Rightarrow \frac{x^2}{(p-x)^2} + \frac{(2x+y)}{(p-x)} + 1 = 0$$

$$\Rightarrow \frac{x^2 + (2x+y)(p-x) + (p-x)^2}{(p-x)^2} = 0$$

$$\Rightarrow \cancel{x^2} + \cancel{2px} - \cancel{2x^2} + py - xy + p^2 + \cancel{x^2} - \cancel{2xp} = 0$$

$$\Rightarrow py - xy + p^2 = 0$$

Ans: $y^2 = cx^2 + \frac{c}{c-1}$.

HW prob 2017 Consider the D.E.,

$$xyp^2 - (x^2+y^2-1)p + xy = 0$$

where $p = \frac{dy}{dx}$. Substituting $u = x^2$ and $v = y^2$, reduce the eqn. to Clairaut's form in terms of u, v and $p' = \frac{dv}{du}$. Hence or otherwise solve the eqn.