KGS IAS

Maths Optional

By Dhrub Singh Sir

$$PI = \frac{1}{D^2-4} 8 \sin 2Z$$

$$= 8 \cdot \frac{1}{D^2-4} \sin 2Z$$

$$= 8 \cdot \frac{1}{-2^2-4} \sin 2Z$$

$$= \frac{\cancel{8}}{-\cancel{8}} \cdot \sin 2Z$$

$$= -\sin 2Z$$

$$= -\sin(x^2)$$

<u>prob:</u> $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

<u>Ans:</u> $y = c_1\cos 2x + c_2 \sin 2x - \cos 2x \cdot \log(\sec 2x + \tan 2x)$

<u>1st HW:</u> <u>Reduced Eqn.</u>

$$\frac{d^2y}{dx^2} - 4y = -8\sin 2Z.$$

$$Z = \frac{x^2}{2}.$$

A.E. $m^2+n^2=0$

$\Rightarrow m=\pm ni$

$y_c = c_1\cos nx + c_2\sin nx$

Let $y_p = A\,u + B\,v$

be the P.I. of (1)

$u=\cos nx \quad v=\sin nx$.

$$W=\begin{vmatrix}\cos nx & \sin nx \\ -n\sin nx & n\cos nx\end{vmatrix}$$

$=n$

Prob: Solve.

$$\frac{d^2y}{dx^2}+n^2y=\sec nx \quad (1)$$

by variation of parameters.

Solⁿ Associated Homo. eqn.

$$\frac{d^2y}{dx^2}+n^2y=0$$

$$(D^2+n^2)y=0$$

$$= \int \frac{\cos nx \cdot \sec nx}{n} dx$$

$$= \frac{1}{n} \int dx = \frac{x}{n}$$

$$\therefore y_p = \frac{1}{n^2} (\log \cos nx) \cos nx + \frac{x}{n} \cdot \sin nx$$

$\therefore y = y_c + y_p$ is the gen. sol^n. of (1).

$$A = \int \frac{-vR}{W} dx$$

$$= \int \frac{-\sin nx \cdot \sec nx}{n} dx$$

$$= +\frac{1}{n} \int \frac{-\sin nx}{\cos nx} dx$$

$$= \frac{1}{n^2} \log \cos nx$$

$$B = \int \frac{uR}{W} dx$$

Sol^n:

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x \quad (1)$$

Comparing.

$$P = -\frac{2(1+x)}{x}, \quad Q = \frac{2(1+x)}{x^2}$$

$$R = x$$

Associated homo. eqn.

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = 0 \quad (ii)$$

2021

prob: Find the general sol^n. of the diff. eqn.

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = 0$$

Hence, solve the D.E

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

by the variation of parameters method.

$$\frac{d^2v}{dx^2}+\left[\frac{-2(\cancel{1}+x)}{x}+\frac{2\cancel{}}{\cancel{x}}\right]\frac{du}{dx}=0$$

$$\Rightarrow \frac{d^2v}{dx^2}+(-2)\frac{du}{dx}=0$$

$$\frac{du}{dx}=p$$

$$\Rightarrow \frac{dp}{dx}-2p=0$$

$$\Rightarrow \frac{dp}{dx}=2p$$

$$\Rightarrow \frac{dp}{p}=2\,dx$$

$$P+Qx$$

$$=\frac{-2(1+x)}{x}+\frac{2(1+x)}{x^2}\times x$$

$$=0$$

$\therefore$ $u=x$ is an integral of CF.

$\therefore$ $y=uv=x\cdot v$ is the gen. solⁿ. of (11), v is given by

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{du}{dx}=0$$

$$v = \frac{c}{2} e^{2x} + c_2$$

$$v = c_1 e^{2x} + c_2$$

$$c_1 = \frac{c}{2}$$

$$\therefore y_c = x\left(c_1 e^{2x} + c_2\right)$$

$$= c_1 \underbrace{x e^{2x}}_{U} + c_2 \underbrace{x}_{v}$$

v sol^n. of (1)

Let $y_p = AU + Bv$ be the P.I.

of (1)

$$W(x) = \begin{vmatrix} xe^{2x} & x \\ e^{2x}+2xe^{2x} & 1 \end{vmatrix}$$

$$\ln p = 2x + \text{Const.}$$

$$\Rightarrow p = e^{2x + \text{const}}$$

$$\Rightarrow p = e^{2x} \cdot \underbrace{e^{\text{const}}}_{c}$$

$$\Rightarrow p = c \cdot e^{2x}$$

$$\Rightarrow \frac{du}{dx} = c e^{2x}$$

$$\Rightarrow du = c e^{2x} dx$$

Int.

$$B = \int \frac{uR}{W}\,dx$$

$$= \int \frac{\cancel{x e^{2x}} \cdot \cancel{x}}{-2\cancel{x^2 e^{2x}}}\,dx$$

$$= -\frac{1}{2}\int dx$$

$$= -\frac{x}{2}$$

$$\therefore y_p = -\frac{1}{4}e^{-2x} \cdot xe^{2x} + \left(-\frac{x}{2}\right)\cdot x$$

$$= -\frac{x}{4} - \frac{x^2}{2}$$

$$\therefore \quad y = y_c + y_p \text{ --- } \cancel{+}1^n \cdot q\,①$$

$$= xe^{2x} - x(e^{2x} + 2xe^{2x})$$

$$= xe^{2x} - xe^{2x} - 2x^2e^{2x}$$

$$= -2x^2e^{2x}.$$

$$A = \int -\frac{vR}{W}\,dx = \int \frac{+\cancel{x^2}}{+2\cancel{x^2}e^{2x}}\,dx$$

$$= \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{1}{4}e^{-2x}.$$

2022 **Prob:** Solve the following diff. eqn. by using the method of variation of parameter:

$$(x^2-1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2-1)^2$$

given that $y = x$ is one soln. of the reduced eqn.

Ans: $y = c_1(x^2+1) + c_2 x - \frac{x^2}{2} + \frac{x^4}{6}$.

(1) can be divided into 4 categories.

(I) Solvable for p.

(II) Solvable for y.

(III) Solvable for x.

(IV) Clairaut's form.

Diff. eqn. of 1st order but not of 1st degree.

$$f(x, y, p) = 0, \quad p = \frac{dy}{dx}.$$

Most general form.

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n y = 0 \quad (1)$$

$P_1, P_2, P_3, \cdots P_n$ are functions of x & y.

$\frac{dy}{dx} = 3$, $\quad \frac{dy}{dx} = 4$

$dy = 3dx$, $\quad dy = 4dx$

Int

$y = 3x + c_1$, $\quad y = 4x + c_2$

$\Rightarrow y - 3x - c_1 = 0$, $\quad y - 4x - c_2 = 0$

Sol^n. is

$$(y - 3x - c)(y - 4x - c) = 0$$

$$c = c_1 = c_2$$

Solvable for p

Prob: Solve

$$p^2 - 7p + 12 = 0$$

Sol^n.

$$p^2 - 3p - 4p + 12 = 0$$

$$\Rightarrow p(p-3) - 4(p-3) = 0$$

$$\Rightarrow (p-3)(p-4) = 0$$

$$\Rightarrow p = 3 \quad \text{or } p = 4$$

$p=0 \Rightarrow \frac{dy}{dx}=0$

$\Rightarrow dy=0$

$\Rightarrow y=c$

$\Rightarrow y-c=0$

$p+2x=0$

$\Rightarrow \frac{dy}{dx}+2x=0$

$\Rightarrow dy+2x\,dx=0$

Int:

$y+x^2=c$

$y+x^2-c=0$

prob: Solve

$$p^3+2xp^2-y^2p^2-2xy^2p=0$$

Solⁿ:

$$p\left[p^2+2xp-y^2p-2xy^2\right]=0$$

$$\Rightarrow p\left[p(p+2x)-y^2(p+2x)\right]=0$$

$$\Rightarrow p(p+2x)(p-y^2)=0$$

$p=0, \quad p+2x=0, \quad p-y^2=0$

Soln: i

$$(y-c)(y+x^2-c)\left(\frac{1}{y}+x+c\right)=0$$

prob: $p^2+2py\cot x=y^2$

Soln:

$$\Rightarrow p^2+2py\cot x-y^2=0$$

$$p=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2 x+4y^2}}{2}$$

$$p=\frac{-\cancel{2}y\cot x\pm\cancel{2}y\operatorname{cosec}x}{\cancel{2}}$$

$$p=y^2$$

$$\Rightarrow \frac{dy}{dx}=y^2$$

$$\Rightarrow \frac{dy}{y^2}=dx$$

$$\Rightarrow y^{-2}dy=dx$$

Int

$$-\frac{1}{y}=x+c$$

$$\Rightarrow \frac{1}{y}+x+c=0$$

$$\log y = \int \frac{-\cos x}{\sin x}\,dx + \int \operatorname{cosec} x\,dx + \text{const}$$

$$= -\log \sin x + \log \tan \tfrac{x}{2} + \text{const}$$

$$\Rightarrow \log y = \log \frac{\tan \frac{x}{2}}{\sin x} + \text{const}$$

$$= \log \frac{\frac{\cancel{\sin \frac{x}{2}}}{\cos \frac{x}{2}}}{2\cancel{\sin \frac{x}{2}} \cdot \cos \frac{x}{2}} + \text{const}$$

$$= \log \frac{1}{2\cos^2 \frac{x}{2}} + \text{const}$$

$$\Rightarrow \log y - \log \frac{1}{1+\cos x} = \text{const}.$$

$$p = -y \cot x \pm y \operatorname{cosec} x.$$

consider

$$p = -y \cot x + y \operatorname{cosec} x$$

$$\Rightarrow \frac{dy}{dx} = -y \cot x + y \operatorname{cosec} x$$

$$\Rightarrow \frac{dy}{y} = -\cot x\,dx + \operatorname{cosec} x\,dx$$

Int.

Gen. soln.

$$\left[y(1+\cos x)-c\right]\left[y(1-\cos x)-c\right]=0$$

Prob: Solve

$$y^2+2xp-x^2p^2=0$$

Ans: $\left[y-c\cdot x^{\frac{1}{2}(1+\sqrt{5})}\right]\left[y-c\cdot x^{\frac{1}{2}(1-\sqrt{5})}\right]=0$

$\Rightarrow \log y(1+\cos x)=\text{const}$

$\Rightarrow y(1+\cos x)=c$

$\Rightarrow y(1+\cos x)-c=0$

Consider

$$p=-y\cot x-y\operatorname{cosec} x$$

Soln

$$y(1-\cos x)=c$$

i.e. $y(1-\cos x)-c=0$

$$2p = \left[-x + 2x^4 p\right]\frac{dp}{dx} + 4x^3p^2$$

$$2p - 4x^3p^2 = -x\left[1 - 2x^3 p\right]\frac{dp}{dx}$$

$$\Rightarrow 2p\left(1 - 2px^3\right) = -x\left[1 - 2px^3\right]\frac{dp}{dx}$$

$$\Rightarrow \underbrace{(1-2px^3)}_{\text{Drop.}}\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\Rightarrow 2p + x\frac{dp}{dx} = 0$$

[Drop
$1 - 2px^3 = 0$
singular soln!]

Solvable for y

prob: Solve
$y + px = p^2x^4$

Soln.

$$\Rightarrow y = -px + p^2x^4 \quad \text{—— (1)}$$

Diff. w.r.t. x

$$p = -\left[p + x\frac{dp}{dx}\right] + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

Eliminating p b/w (I) & (II),

$$y = -\left(\frac{c}{x}\right)^2 \cdot x + \left(\frac{c}{x}\right)^4 \cdot x^4$$

$$\boxed{y = -\frac{c^2}{x} + c^4}$$

$$y + \frac{c_1}{x} = c_1^2$$

$c_1 = c^2$ say

$$\Rightarrow 2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow 2p\,dx + x\,dp = 0$$

$$\Rightarrow \frac{dx}{x} + \frac{dp}{2p} = 0$$

Int

$$\log x + \frac{1}{2}\log p = \text{const}$$

$$\Rightarrow \log x \cdot \sqrt{p} = \text{const}.$$

$$\Rightarrow x \cdot \sqrt{p} = c \quad \text{—— (II)}$$

Prob: Solve

$$x - yp = ap^2$$

Soln: $yp = x - ap^2$

$$\Rightarrow y = \frac{x}{p} - ap \quad \text{--- (1)}$$