Maths Optional

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For x rational

$$|f(x) - f(0)| = |x - 0|$$
$$= |x|$$

$|f(x) - f(0)| < \varepsilon$ when $|x| < \varepsilon$

Choose $\delta = \varepsilon$

$$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$ (1)

HW prob:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rat.} \\ 0, & \text{if } x \text{ is irr} \end{cases}$$

→ Case (iii) $x = 0$

Let $\varepsilon > 0$ be given.

For x irrational.

$$|f(x) - f(0)| = |0 - 0|$$
$$= 0 < \varepsilon. \rightarrow (2)$$

$\rightarrow$ Let $\alpha = x^2$

$$\lim_{n\to\infty} \frac{\alpha^n}{1+\alpha^n} = \begin{cases} 0, & 0 \le \alpha < 1 \\ \frac{1}{2}, & \alpha = 1 \\ 1, & \alpha > 1 \end{cases}$$

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ \frac{1}{2}, & x = \pm 1 \\ 1, & x < -1 \text{ or } x > 1 \end{cases}$$

Check cont. at $x = \pm 1$ (do it).

From (I) & (II)

For any real x,

$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$

$\therefore f$ is cts at $x = 0$

<u>Prob:</u> Show that

$f(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}}$ is cts

at all points of $\mathbb{R}$ except

$x = \pm 1$.

$\forall n \in \mathbb{N}$,

$$f(a_n) = a_n + 3$$

$$\Rightarrow \lim f(a_n) = \lim (a_n + 3)$$

$$= \lim a_n + 3$$

$$= a + 3$$

$$\neq 2a = f(a)$$

$\therefore \langle f(a_n) \rangle \not\to f(a)$

$\therefore$ fn is not cts at $x = a$.

HW

$$f(x) = \begin{cases} 2x, & \text{if } x \text{ is rational} \\ x+3, & \text{if } x \text{ is irr.} \end{cases}$$

$\longrightarrow$ Case 1 a is a rational no. $(a \neq 3)$

$$f(a) = 2a.$$

we have a seq. of irrational number $\langle a_n \rangle$ conv. to 'a'.

choose $\delta = \varepsilon$

so, $|f(x) - f(c)| < \varepsilon$

when $|x - c| < \delta$

$\therefore$ f is cts at $x = c$.

$\because$ c is any ~~point~~ point of $\mathbb{R}$.

$\therefore$ f is cts in $\mathbb{R}$.

<u>Prob</u>: Show that $f(x) = |x|$ is cts in $\mathbb{R}$.

$\rightarrow$ Let $c \in \mathbb{R}$ be any no.

Let $\varepsilon > 0$ be given.

$$|f(x) - f(c)| = \big||x| - |c|\big|$$

$$\leq |x - c|$$

$\therefore |f(x) - f(c)| < \varepsilon$ when $|x - c| < \varepsilon$

$$|f(x)-f(c)| \le K|x-c| < \varepsilon$$

if $|x-c| < \frac{\varepsilon}{K}$

choose $\delta = \frac{\varepsilon}{K}$

$\therefore$ For $\varepsilon > 0$, $\exists$ some $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$$

$\therefore$ f is cts at $x = c$.

$\because$ c is any point in $\mathbb{R}$.

$\therefore$ f is cts in $\mathbb{R}$.

<u>Prob</u>: Let $K > 0$ and $f: \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x)-f(y)| \le K|x-y| \quad \text{--- (1)}$$

$\forall x, y \in \mathbb{R}$. Show that f is cts in $\mathbb{R}$.

$\rightarrow$ Let c be any real no.

Let $\varepsilon > 0$ be given

Put $y = c$ in (1)

→ Case ① Let a be a rational no. $a \in]0,1[$

Let $a = \frac{p}{q}$, p, q are +ve int & having no common factor.

$\therefore f(a) = \frac{1}{q} > 0$

Also, $\exists$ some seq. of irrational no. $\langle a_n \rangle$ s.t. $\langle a_n \rangle \to a$.

Prob: Let f be a function defined on $]0,1[$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational.} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, \text{ where } p \text{ \& } q \text{ are +ve int. having no common factor.} \end{cases}$$

Prove that f is cts at each irrational pt. and disc. at each rational pt.

$\therefore$ By Archemedian prop.

of real no. $\exists$ a +ve int.

n s.t.

$$n > \frac{1}{\varepsilon}$$

i.e. $\frac{1}{n} < \varepsilon$.

It is clear that there are finitely many $\frac{p}{q} \in]0,1[$

s.t. $q < n$.

For each $n \in \mathbb{N}$,

$$f(a_n) = 0$$

$$\therefore \langle f(a_n) \rangle \to 0 \neq \frac{1}{q} = f(a)$$

> 0

$\therefore$ f is not cts at $x = a$.

Case (ii) Let $b \in]0,1[$ be an irrational no.

$$\therefore f(b) = 0$$

Let $\varepsilon > 0$ be given.

and for $x = \frac{l}{m}$ (rational)

$|x-b| < \delta$

$\Rightarrow |f(x) - f(b)|$

$= \left|\frac{1}{m} - 0\right|$

$= \frac{1}{m} \le \frac{1}{n} < \varepsilon$ — (ii)

From (i) & (ii)

$|x-b| < \delta \Rightarrow |f(x) - f(b)| < \varepsilon$

$\therefore$ f is cts at $x = b$. x is real

$\therefore$ we can find some $\delta > 0$

s.t $]b-\delta, b+\delta[$

contains no rational

no. $\frac{l}{m}$ s.t $m < n$

$\therefore m \ge n$

i.e. $\frac{1}{m} \le \frac{1}{n}$.

Thus

$|x-b| < \delta \Rightarrow |f(x) - f(b)|$

$= |0-0| = 0 < \varepsilon$ — (i)

when x is irr.

and let $f(x) = x+1 \ \forall x \in \mathbb{R}$.
Show that $g \circ f$ is not cts at $x = 0$.

$\longrightarrow$

$$g \circ f(x) = g(f(x))$$

$$= g(x+1) = \begin{cases} 0, & \text{if } x+1 = 1 \\ 2, & \text{if } x+1 \neq 1 \end{cases}$$

i.e., $g \circ f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 2, & \text{if } x \neq 0 \end{cases}$

Result: Let f & g be defined on I & J resp. and let $f(I) \subseteq J$. If f is cts at $p \in I$ and g is cts at $f(p) \in J$ then $g \circ f$ is cts at 'p'.

Proof: Let g be defined on $\mathbb{R}$ by

$$g(x) = \begin{cases} 0, & \text{if } x = 1 \\ 2, & \text{if } x \neq 1. \end{cases}$$

Result: If a function f is cts at $x=c$ then $|f|$ is also cts at $x=c$. Converse is not true.

$\rightarrow$ Let $\varepsilon>0$ be given.

$\because$ f is cts at $x=c$.

$\therefore$ Corr. to $\varepsilon>0$, $\exists$ some $\delta>0$

s.t. $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

(1)

$$\lim_{x\to 0} g\circ f(x)$$

$$= \lim_{x\to 0} 2 = 2$$

$g\circ f(0) = 0$

$\therefore \lim_{x\to 0} g\circ f(x) \neq g\circ f(0)$

$\therefore$ $g\circ f$ is not cts at $x=0$

Converse is not true.

Consider

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

f is not cts at $x = 0$

$$|f|(x) = 1 \quad \forall x \in \mathbb{R}$$

Here, $|f|$ is cts at $x = 0$

Now

$$\big| |f|(x) - |f|(c) \big|$$

$$= \big| |f(x)| - |f(c)| \big|$$

$$\leq |f(x) - f(c)|$$

$$< \varepsilon \quad \text{when } |x - c| < \delta$$

(using (1))

$\therefore |f|$ is cts at $x = c$.

$\rightarrow$

Case ① $f(x) \geq g(x) \quad \forall x \in \mathbb{R}$

$$h(x) = \max\{f(x), g(x)\}$$
$$= f(x)$$

$$\frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$$
$$= \frac{1}{2}(f(x) + \cancel{g(x)}) + \frac{1}{2}(f(x) - \cancel{g(x)})$$
$$= f(x)$$

$$\therefore h(x) = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g|$$

Result: Let $f, g : \mathbb{R} \to \mathbb{R}$ be cts at 'c'. And let

$$h(x) = \max\{f(x), g(x)\} \quad \forall x \in \mathbb{R}$$
or
$$\sup\{f(x), g(x)\}$$

Show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ $\forall x \in \mathbb{R}$. Use this to show that $h(x)$ is cts at 'c'.

$\therefore h(x) = \frac{1}{2}\left(f(x)+g(x)\right) + \frac{1}{2}\left|f(x)-g(x)\right|$

$\because$ f and g are cts at 'c'.

$\therefore$ f + g & f − g are also cts at 'c'.

$\therefore$ f + g & |f − g| are also cts at 'c'.

$\therefore \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ ''

$\therefore$ h(x) is cts at 'c'.

Case (II) $f(x) < g(x) \quad \forall x \in R$

$h(x) = \max\{f(x), g(x)\}$

$= g(x)$

$\frac{1}{2}\left(f(x)+g(x)\right) + \frac{1}{2}\left|f(x)-g(x)\right|$

$= \frac{1}{2}\left(\cancel{f(x)}+g(x)\right) + \frac{1}{2}\left[-\left(\cancel{f(x)}-g(x)\right)\right]$

$= g(x).$

Result: If a function f is cts at $x=c$ then it is bdd in some nbd of 'c'.

$\rightarrow$ Let $\varepsilon = \frac{1}{2} > 0$

$\because$ f is cts at $x=c$.

$\therefore$ Corr. to $\varepsilon = \frac{1}{2}$, $\exists$ some $\delta > 0$ s.t.

$|f(x) - f(c)| < \frac{1}{2}$, when $|x-c| < \delta$

HW **Result:** Let $f, g : \mathbb{R} \to \mathbb{R}$ be cts at 'c' and let

$h(x) = \min\{f(x), g(x)\}$

or

$\inf\{f(x), g(x)\}$ $\forall x \in \mathbb{R}$

Show that $h(x) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)|$ $\forall x \in \mathbb{R}$. Use this to show that $h(x)$ is cts at $x = c$.

Boundedness of continuous fn.

A function $f:[a,b] \to \mathbb{R}$ is said to be bdd in $[a,b]$ if there exist real no. k and K s.t.

$$k \le f(x) \le K \quad \forall x \in [a,b]$$

Ex ① $f(x) = x^2$ is bdd in $[1,2]$

$$1 \le f(x) \le 4 \quad \forall x \in [1,2]$$

$\therefore$

$$f(c) - \tfrac{1}{2} < f(x) < f(c) + \tfrac{1}{2}$$

when $x \in \,]c-\delta, c+\delta[$

Let $k = f(c) - \frac{1}{2}$

$K = f(c) + \frac{1}{2}$

$\therefore$

$$k < f(x) < K \quad \forall x \in \,]c-\delta, c+\delta[$$

$\therefore$ $f(x)$ is bdd in some nbd. $]c-\delta, c+\delta[$ of 'c'.

Ex (11) Is $f(x) = \frac{1}{x-1}$ bdd in $[1, 4]$?

It is unbounded.

As it is not bdd above.