KGS IAS

Maths Optional

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Cauchy sequence: A seq. $\langle a_n \rangle$ is said to be a cauchy seq. if for any $\varepsilon > 0$, $\exists$ a +ve integer m s.t.

$$|a_n - a_m| < \varepsilon \quad \forall n \geq m$$

or

$$|a_{n+p} - a_n| < \varepsilon \quad \forall n \geq m \ \& \ p \geq 1.$$

Ex: $\langle 1/n \rangle$ is a Cauchy seq.

Let $a_n = 1/n$

Let $\varepsilon > 0$ be given.

For $n > m$

$$|a_n - a_m| = \left| \frac{1}{n} - \frac{1}{m} \right| = -\left(\frac{1}{n} - \frac{1}{m}\right)$$

$$= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}$$

Ex: The seq. $\left\langle \frac{n}{n+1} \right\rangle$ is a Cauchy seq.

$\rightarrow$ Let $a_n = \frac{n}{n+1}$.

Let $\varepsilon > 0$ be given.

For $n > m$

$$|a_n - a_m|$$

$$= \left| \frac{n}{n+1} - \frac{m}{m+1} \right|$$

$$|a_n - a_m| < \frac{1}{m} < \varepsilon$$

if $m > \frac{1}{\varepsilon}$

Let m be a +ve int. greater than $\frac{1}{\varepsilon}$.

$\therefore |a_n - a_m| < \varepsilon \quad \forall n \geq m$

$\therefore \langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle$ is a Cauchy seq.

$$|a_n - a_m| = \frac{n-m}{n+1} \cdot \frac{1}{m+1}$$

$$< \frac{1}{m+1} \qquad \left[\because \frac{n-m}{n+1} < 1\right]$$

$$< \varepsilon \text{ if } m+1 > \frac{1}{\varepsilon}$$

$$\text{i.e. if } m > \frac{1}{\varepsilon} - 1$$

Let m be a +ve int. $> \frac{1}{\varepsilon} - 1$.

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall\, n \geq m$.

$\therefore \langle a_n \rangle \equiv \left\langle \frac{n}{n+1} \right\rangle$ is Cauchy.

$$= \left| \frac{n(m+1) - m(n+1)}{(n+1)(m+1)} \right|$$

$$= \left| \frac{\cancel{nm} + n - \cancel{mn} - m}{(n+1)(m+1)} \right|$$

$$= \left| \frac{n-m}{(n+1)(m+1)} \right|$$

$$= \frac{n-m}{(n+1)(m+1)}$$

$|a_n - a_m| < 1 \quad \forall n \geq m$ — (1)

If m is even, $a_m = 1$

Let $n = \underbrace{2m+1} > m$

$a_{2m+1} = -1$

$|a_n - a_m| = |-1-1|$

$= 2 > 1$

Contradiction to (1)

<u>Ex:</u> The seq. $\langle (-1)^n \rangle$ is not a Cauchy seq.

→ Let, if ~~possible~~,

$\langle (-1)^n \rangle$ is a Cauchy seq.

Let $a_n = (-1)^n$

Choose $\varepsilon = 1$

$\because \langle a_n \rangle$ is Cauchy.

$\therefore \exists$ a +ve int. m s.t.

$\therefore$ our supposition is wrong.

$\therefore$ $\langle (-1)^n \rangle$ is not a Cauchy seq.

<u>Th:</u> prove that every Cauchy seq. is ~~bounded~~. Converse is not true.

<u>Proof:</u> Let $\langle a_n \rangle$ is a Cauchy seq.

Choose $\varepsilon = \frac{1}{2}$

If m is odd, $a_m = -1$

Let $n = 2m > m$

$\therefore a_n = a_{2m} = 1$

Now $|a_n - a_m| = |1 - (-1)|$

$= |2|$

$= 2 > 1$

Contra. to ①

$\therefore k \le a_n \le K \ \forall n.$

$\therefore \langle a_n \rangle$ is bdd.

Converse is not true.

Consider the example

$$\langle (-1)^n \rangle \text{ — bdd.}$$

But not Cauchy.

$\because \langle a_n \rangle$ is Cauchy.

$\therefore$ Corr. to $\varepsilon = \frac{1}{2}$, $\exists$ a +ve int. m s.t.

$$|a_n - a_m| < \tfrac{1}{2} \ \forall n \ge m$$

$$\Rightarrow a_m - \tfrac{1}{2} < a_n < a_m + \tfrac{1}{2} \ \forall n \ge m$$

Let $k = \min\{a_1, a_2, \ldots a_{m-1}, a_m - \frac{1}{2}\}$

Let $K = \max\{a_1, a_2, \ldots a_{m-1}, a_m + \frac{1}{2}\}$.

$\therefore$ corr. to $\frac{\varepsilon}{2} > 0$, $\exists$ a +ve int. m s.t.

$$|a_n - l| < \frac{\varepsilon}{2} \quad \forall n \geq m \text{ --- } (1)$$

In particular,

(1) is true for $n = m$

$$|a_m - l| < \frac{\varepsilon}{2}$$

Now
$$\begin{aligned}|a_n - a_m| &= |a_n - l + l - a_m| \\ &< |a_n - l| + |l - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Cauchy's general principle of convergence:

Stat: A seq. converges iff it is a Cauchy seq.

Proof: Let $\langle a_n \rangle$ be a cgt. seq. & let $\lim a_n = l$.

Let $\varepsilon > 0$ be given

$\because \langle a_n \rangle \to l$

$\therefore$ By BWT for seq.

$\langle a_n \rangle$ has a limit-point, p (say).

<u>Claim</u>: $\lim a_n = p$.

Let $\varepsilon > 0$ be given

$\because \langle a_n \rangle$ is a Cauchy seq.

$\therefore$ corr. to $\frac{\varepsilon}{3} > 0$, $\exists$ a

$\forall\, n \geq m$

$\therefore$ corr. to $\varepsilon > 0$, we have found a +ve int. m s.t.

$|a_n - a_m| < \varepsilon \quad \forall\, n \geq m$

$\therefore \langle a_n \rangle$ is Cauchy.

<u>Conversely</u> Suppose that $\langle a_n \rangle$ is a Cauchy seq.

$\therefore$ It is bdd.

i.e.

$$p - \varepsilon/3 < a_K < p + \varepsilon/3$$

$$\Rightarrow |a_K - p| < \varepsilon/3 \quad \text{—— (III)}$$

$\because K > m \quad \therefore$ (II) holds for $n = K$ also.

i.e. $|a_K - a_m| < \varepsilon/3$ —— (IV)

$$|a_n - p| = |(a_n - a_m) + (a_m - a_K) + (a_K - p)|$$

+ve int. m s.t.

$$|a_n - a_m| < \varepsilon/3 \quad \forall n \geq m \quad \text{—— (II)}$$

$\because p$ is a limit ~~point~~ of $\langle a_n \rangle$

$\therefore \exists$ a +ve int $K > m$ s.t.

$$a_K \in \left]p - \varepsilon/3, p + \varepsilon/3\right[$$

$|a_n - p| < \varepsilon \quad \forall n \geq m$

$\therefore \langle a_n \rangle \to p.$

<u>prob:</u> Show that the seq. $\langle a_n \rangle$, where

$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

is not convergent.

$\rightarrow$ Let, if ~~possible~~, $\langle a_n \rangle$ is convergent.

$\leq |a_n - a_m| + |a_m - a_k| + |a_k - p|$

$= |a_n - a_m| + |a_k - a_m| + |a_k - p|$

$< \underbrace{\varepsilon/3 + \varepsilon/3 + \varepsilon/3}_{=\varepsilon} \quad \left[\because |a_m - a_k| = |a_k - a_m|\right]$

$\forall n \geq m$

[using (ii), (iii) & (iv)]

$\therefore$ for any $\varepsilon > 0$ $\exists$ a +ve int m s.t.

select:

$n = 2m > m$

$\therefore |a_n - a_m|$

$= |a_{2m} - a_m|$

$= \left| 1 + \cancel{\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}} + \frac{1}{m+1} + \cdots + \frac{1}{2m} - \left(1 + \cancel{\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}}\right) \right|$

$= \left| \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \right|$

$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$

$> \frac{1}{2m} + \frac{1}{2m} + \cdots$ m times.

$= m \cdot \frac{1}{2m} = \frac{1}{2}$

$\therefore$ By Cauchy's convergence criterion, $\langle a_n \rangle$ is a Cauchy seq.

Choose $\varepsilon = \frac{1}{2}$

$\because \langle a_n \rangle$ is Cauchy.

$\therefore$ corr. to $\varepsilon = \frac{1}{2}$, $\exists$ a +ve int m s.t.

$|a_n - a_m| < \frac{1}{2} \quad \forall n \geq m$ — (1)

Prob: Show that the seq. $\langle a_n \rangle$ defined by

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

does not converge.

→ Let, if possible, $\langle a_n \rangle$ is convergent.

$\therefore$ It is Cauchy.

Choose $\varepsilon = \frac{1}{4}$

$\therefore |a_{2m} - a_m| > \frac{1}{2}$

which is a contradiction to ①.

$\therefore$ our supposition is wrong.

$\therefore \langle a_n \rangle$ is not cgt.

$$\left|1+\frac{1}{3}+\cdots+\frac{1}{2m-1}+\frac{1}{2m+1}+\cdots+\frac{1}{4m-1}-\left(1+\frac{1}{3}+\cdots+\frac{1}{2m-1}\right)\right|$$

$$=\left|\frac{1}{2m+1}+\frac{1}{2m+3}+\cdots+\frac{1}{4m-1}\right|$$

$$=\frac{1}{2m+1}+\frac{1}{2m+3}+\cdots+\frac{1}{4m-1}$$

$$>\frac{1}{4m}+\frac{1}{4m}+\cdots \; m \text{ times}$$

$$=m\cdot\frac{1}{4m}=\frac{1}{4}$$

$\because \langle a_n \rangle$ is Cauchy.

$\therefore$ corr. to $\varepsilon=\frac{1}{4}$, $\exists$ a +ve int. m s.t.

$$|a_n-a_m|<\frac{1}{4} \quad \text{——— (1)}$$

$$\forall\, n \geq m$$

$n=2m>m$

Now $|a_n-a_m|=|a_{2m}-a_m|$

$\rightarrow \quad \varepsilon = \frac{1}{6}$

$$|a_{2m} - a_m|$$
$$= \frac{1}{3m+1} + \frac{1}{3m+4} + \cdots \frac{1}{6m-2}$$
$$> \frac{1}{6m} + \frac{1}{6m} + \cdots \text{ m times}$$
$$= m \cdot \frac{1}{6m} = \frac{1}{6}$$

$\therefore |a_{2m} - a_m| > \frac{1}{4}$

Contradiction to ①

$\therefore$ our supposition is wrong.

$\therefore \langle a_n \rangle$ is not cgt.

Prob: Show that the seq. $\langle a_n \rangle$ defined by

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3n-2}$$

is not cgt.

Monotonic sequences

→ A seq. $\langle a_n \rangle$ is said to be monotonically increasing seq. iff $a_{n+1} \geq a_n \ \forall n$.

i.e. $a_1 \leq a_2 \leq a_3 \leq a_4 \cdots$

Equivalently, A seq. $\langle a_n \rangle$ is said to be monotonically increasing seq iff

$$a_n \geq a_m \quad \forall n \geq m$$

→ A seq. $\langle a_n \rangle$ is said to be monotonically decreasing seq. iff

$$a_{n+1} \leq a_n \quad \forall n$$

i.e. $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots$

Equivalently, a seq. $\langle a_n \rangle$ is said to be monotonically decreasing seq. iff

$$a_n \leq a_m \quad \forall n \geq m.$$

(ii) $\langle \frac{1}{n} \rangle$

$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$

Monotonically $\downarrow$ seq.

(iii) $\langle (-1)^n \rangle$

$\langle -1, 1, -1, 1, \ldots \rangle$

Not monotonic.

$\rightarrow$ A seq. is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Ex: (1) $\langle n^2 \rangle$

$\langle 1^2, 2^2, 3^2, \ldots \rangle$

Monotonically $\uparrow$ seq.

Conversely $\langle a_n \rangle$ is monotonic and is bdd.

To prove: $\langle a_n \rangle$ is cgt.

Suppose $\langle a_n \rangle$ is a monotonically $\uparrow$ seq.

Let $S = \{a_n : n \in \mathbb{N}\}$ be its range set.

$\therefore \langle a_n \rangle$ is bdd.

Th: A necessary and sufficient condition for the convergence of a monotonic seq. is that it is bounded.

Proof: Suppose $\langle a_n \rangle$ is a monotonic cgt. seq.

$\therefore \langle a_n \rangle$ is cgt.

$\therefore$ It is bdd.

$\therefore$ $p-\varepsilon$ can not be an upper bound of S.

$\therefore$ $\exists$ some $a_m \in S$

s.t. $p-\varepsilon < a_m$

$\because$ $\langle a_n \rangle$ is monotonically increasing seq.

$\therefore$ $a_n \geq a_m > p-\varepsilon$ $\forall n \geq m$

$\therefore$ $a_n > p-\varepsilon$ $\forall n \geq m$ — ①

$\therefore$ S is also bounded.

By order-completeness property of real number, S has a l.u.b (supremum)

Let $\sup S = p$.

<u>Claim:</u> $\lim a_n = p$.

Let $\varepsilon > 0$ be given

$\therefore$ $p-\varepsilon < p$

Similarly, we can prove it if we consider $\langle a_n \rangle$ as a monotonically decreasing seq.

$\therefore \sup S = p$

$\therefore a_n \leq p < p + \varepsilon$ ——— (ii) $\forall n$

(i) & (ii) $\Rightarrow$

$p - \varepsilon < a_n < p + \varepsilon \quad \forall n \geq m$

$\Rightarrow |a_n - p| < \varepsilon \quad \forall n \geq m.$

$\therefore \lim a_n = p.$