KGS IAS

Maths Optional

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$$= 1+1+\frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right)+\frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+ \cdots + \frac{1}{\lfloor n}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \quad \text{— (1)}$$

$$s_{n+1} = 1+1+\frac{1}{\lfloor 2}\left(1-\frac{1}{n+1}\right)$$

$$+\frac{1}{\lfloor 3}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots$$

$$+\frac{1}{\lfloor n+1}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \quad \text{— (11)}$$

<u>Prob:</u> Show that the sequence $\langle s_n \rangle$, where $s_n = \left(1+\frac{1}{n}\right)^n$ is convergent and that limit s_n lies b/w 2 & 3.

$\rightarrow$ $s_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{\lfloor 2}\left(\frac{1}{n}\right)^2$$

$$+ \cdots + \frac{n(n-1)\cdots(n-(n-1))}{\lfloor n}\cdot\left(\frac{1}{n}\right)^n$$

From (i) & (ii)

$s_{n+1} > s_n \quad \forall n.$

$\therefore \langle s_n \rangle$ is monotonically increasing seq.

$s_n = \underline{1+1} + \frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{\lfloor n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{n-1}{n}\right)$

$s_n > 2 \quad \forall n.$

$n+1 > n$

$\Rightarrow \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow -\frac{1}{n+1} > -\frac{1}{n}$

$\Rightarrow 1-\frac{1}{n+1} > 1-\frac{1}{n}$

IIly, we can show that

$1-\frac{2}{n+1} > 1-\frac{2}{n}$

$1-\frac{3}{n+1} > 1-\frac{3}{n}$ & so on.

$$s_n < 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 1 + 2 - \frac{1}{2^{n-1}}$$

$$\underline{s_n} < 3 - \frac{1}{2^{n-1}} < 3$$

$\forall n.$

$\therefore\ 2 < s_n < 3 \quad \forall n.$

$\therefore \langle s_n \rangle$ is monot. $\uparrow$ and is bdd above.

$\therefore$ s_n is cgt.

$$s_n = \underline{1 + 1} + \frac{1}{\lfloor 2}\left(1 - \frac{1}{n}\right) + \frac{1}{\lfloor 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{\lfloor n}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

Prob: Show that the seq. $\langle a_n \rangle$ defined by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), \; n \geq 1$$

& $a_1 > 0$; converge to 3.

$\rightarrow \because a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$

$\Rightarrow a_{n+1} = \frac{1}{2}\left[\frac{a_n^2 + 9}{a_n}\right]$

$\therefore 2 < s_n < 3 \quad \forall n.$

$\therefore 2 < \underbrace{\lim s_n}_{e} < 3$

$$\boxed{2 < e < 3}$$

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$$

$a_{n+1}^2 \geq 9$

$\Rightarrow a_{n+1} \geq 3 \quad \left[\because a_n > 0\right]$

$\forall n.$

$\therefore \langle a_n \rangle$ is bdd below.

Now,

$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$

$= a_n - \frac{1}{2}\left[\frac{a_n^2 + 9}{a_n}\right]$

$2 a_n a_{n+1} = a_n^2 + 9$

$\Rightarrow a_n^2 - 2a_{n+1} a_n + 9 = 0$

quadratic in a_n.

It has real roots

if

$(-2a_{n+1})^2 - 4 \times 1 \times 9 \geq 0$

$\Rightarrow \cancel{4} a_{n+1}^2 \geq \cancel{36}\ 9$

$\therefore \langle a_n \rangle$ is mono. decreasing.

Also it is bdd below.

$\therefore$ it is cgt

Let $\lim a_n = l$.

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$$

$$\Rightarrow \lim a_{n+1} = \lim \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$$

$$\Rightarrow l = \frac{1}{2}\left(l + \frac{9}{l}\right)$$

$$= \frac{2a_n^2 - a_n^2 - 9}{2a_n}$$

$$= \frac{a_n^2 - 9}{2a_n} \geq 0$$

$$\therefore a_n - a_{n+1} \geq 0 \quad \forall n$$

$$\Rightarrow a_n \geq a_{n+1}$$

$$\Rightarrow a_{n+1} \leq a_n \quad \forall n$$

prob: If $\langle a_n \rangle$ is a bounded seq. such that

$$s_{n+1} = \sqrt{\frac{ab^2 + s_n^2}{a+1}}, \; b > a$$

$\forall n \geq 1$, $s_1 = a > 0$ then show that the seq. $\langle s_n \rangle$ is an increasing seq. and $\lim_{n\to\infty} s_n = b$.

$$2l = l + 9/l$$

$$\Rightarrow 2l = \frac{l^2+9}{l}$$

$$\Rightarrow 2l^2 = l^2 + 9$$

$$\Rightarrow l^2 = 9$$

$$\Rightarrow l = \pm 3$$

$$(l \neq -3)$$ $a_n > 0$ $\forall n.$

$$\therefore l = 3$$

$s_2^2 - s_1^2 > 0$

$\Rightarrow s_2^2 > s_1^2$

$\Rightarrow s_2 > s_1 \quad \left[\because \begin{array}{l} s_1 > 0 \\ s_2 > 0 \end{array}\right]$

Suppose,

$s_{k+1} > s_k$

$\Rightarrow s_{k+1}^2 > s_k^2$

$\Rightarrow ab^2 + s_{k+1}^2 > ab^2 + s_k^2$

$$s_2^2 - s_1^2 = \frac{ab^2 + s_1^2}{a+1} - s_1^2$$

$$= \frac{ab^2 + a^2}{a+1} - a^2$$

$$= \frac{ab^2 + \cancel{a^2} - a^3 - \cancel{a^2}}{a+1}$$

$$= \frac{a(b^2 - a^2)}{a+1} > 0$$

$\left[\begin{array}{l} b > a \\ \Rightarrow b^2 > a^2 \end{array}\right]$

$\therefore \langle s_n \rangle$ is mono. increasing seq.

And also it is bdd.

$\therefore$ it is cgt.

let $\lim s_n = l$

$$s_{n+1}^2 = \frac{ab^2 + s_n^2}{a+1}$$

$$\Rightarrow \lim s_{n+1}^2 = \lim \frac{ab^2 + s_n^2}{a+1}$$

$$\Rightarrow \frac{ab^2 + s_{k+1}^2}{a+1} > \frac{ab^2 + s_k^2}{a+1}$$

$[a > 0]$

$$\Rightarrow \sqrt{\frac{ab^2 + s_{k+1}^2}{a+1}} > \sqrt{\frac{ab^2 + s_k^2}{a+1}}$$

$$\Rightarrow s_{k+2} > s_{k+1}$$

$\therefore$ By PMI,

$s_{n+1} > s_n \quad \forall n.$

$l \neq -b \quad [\because s_n > 0 \ \forall n]$

$\therefore l = b.$

i.e. $\lim s_n = b$

$$l^2 = \frac{a \cdot b^2 + l^2}{a+1}$$

$$\Rightarrow a l^2 + \cancel{l^2} = a \cdot b^2 + \cancel{l^2}$$

$$\Rightarrow a l^2 - a b^2 = 0$$

$$\Rightarrow a(l^2 - b^2) = 0$$

$$\Rightarrow l^2 - b^2 = 0$$

$$\Rightarrow l = \pm b.$$

Note: The condition $i > j \Rightarrow n_i > n_j$ guarantees that the order of the terms of a subsequence is the same as that of the sequence.

Ex: $\langle 1^2, 3^2, 5^2, \dots (2n-1)^2 \dots \rangle$ is a subsequence of $\langle n^2 \rangle$.

Subsequence: Let $\langle a_n \rangle$ be a sequence. Let $\langle n_1, n_2, n_3, \dots \rangle$ be a seq. of +ve integers such that $i > j \Rightarrow n_i > n_j$. Then the sequence $\langle a_{n_1}, a_{n_2}, a_{n_3}, \dots \rangle$, written as $\langle a_{n_k} \rangle$ is called a subsequence of $\langle a_n \rangle$.

$\therefore$ corr. to $\varepsilon > 0$, $\exists$ a +ve int. m s.t.

$$|a_n - l| < \varepsilon \quad \forall n \geq m$$

In particular,

$$|a_{n_k} - l| < \varepsilon, \quad \forall n_k \geq m$$

$$\therefore \langle a_{n_k} \rangle \longrightarrow l.$$

Converse is not true.

<u>Result</u>: If a sequence $\langle a_n \rangle$ converges to l, then every subsequence of $\langle a_n \rangle$ converges to l. Converse is not true.

<u>Proof</u>: Let $\langle a_{n_k} \rangle$ be a subsequence of $\langle a_n \rangle$.

Let $\varepsilon > 0$ be given.

$$\because \langle a_n \rangle \longrightarrow l.$$

Result: A real number l is a limit point of a seq. $\langle a_n \rangle$ if and only if there exist a subsequence of $\langle a_n \rangle$ converging to l.

Cor: Every bounded seq. has a convergent sub-sequence.

Consider the example

$$\langle (-1)^n \rangle$$

$$\langle -1, 1, -1, 1, -1, 1, \ldots \rangle$$

$$\langle 1, 1, 1, 1, \ldots \rangle \to 1$$

But $\langle (-1)^n \rangle$ is not cgt.

If $s_1 = s_2$

$$s_3 = \frac{1}{2}(s_2 + s_1)$$

$$= \frac{1}{2}(s_1 + s_1) = s_1$$

$$s_4 = \frac{1}{2}(s_3 + s_2)$$

$$= \frac{1}{2}(s_1 + s_1) = s_1$$

$$s_n = s_1 \quad \forall n.$$

$$\lim s_n = s_1$$

Prob: If $\langle s_n \rangle$ be a sequence of positive numbers such that $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$ $\forall n > 2$ then show that $\langle s_n \rangle$ converges. Also find $\lim s_n$.

$\rightarrow$ $$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}) \quad \text{——} (1)$$

$s_5 = \frac{1}{2}(s_4 + s_3)$

$\Rightarrow s_5$ is the A.M. of s_3 & s_4

$\therefore s_3 < s_5 < s_4$

and so on.

$\underbrace{s_1 < s_3 < s_5}_{\uparrow} < \cdots \underbrace{< s_6 < s_4}_{\searrow} < s_2$

$(s_1 \neq s_2)$

If $s_1 < s_2$,

$s_3 = \frac{1}{2}(s_2 + s_1)$

$\Rightarrow s_3$ is the A.M. of s_1 & s_2

$\therefore s_1 < s_3 < s_2$

$s_4 = \frac{1}{2}(s_3 + s_2)$

$\Rightarrow s_4$ is the A.M. of s_3 & s_2

$\therefore s_3 < s_4 < s_2$

$(1) \Rightarrow s_{2n} = \frac{1}{2}(s_{2n-1} + s_{2n-2})$

$\Rightarrow \lim s_{2n} = \lim \frac{1}{2}(s_{2n-1} + s_{2n-2})$

$\Rightarrow l' = \frac{1}{2}(l + l')$

$\Rightarrow 2l' = l + l'$

$\Rightarrow l' = l$.

$\therefore \langle s_{2n} \rangle$ and $\langle s_{2n-1} \rangle$ Converge to the same limit l.

$\therefore \langle s_{2n-1} \rangle$ is mono. incr. and is bdd above by s_2

& $\langle s_{2n} \rangle$ is mono. decr. and is bdd below by s_1.

Both the subsequences are cgt.

Suppose

$\lim s_{2n-1} = l$ and $\lim s_{2n} = l'$.

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$s_n + \frac{1}{2} s_{n-1} = \cdots = \frac{1}{2}(s_1 + 2s_2)$$

i.e. $s_n + \frac{1}{2} s_{n-1} = \frac{1}{2}(s_1 + 2s_2)$

$$\Rightarrow \lim \left(s_n + \frac{1}{2} s_{n-1}\right) = \frac{1}{2}(s_1 + 2s_2)$$

$$\Rightarrow \left(l + \frac{1}{2} l\right) = \frac{1}{2}(s_1 + 2s_2)$$

$$\Rightarrow \frac{3l}{\cancel{2}} = \frac{1}{\cancel{2}}(s_1 + 2s_2)$$

$\therefore \lim s_n = l.$

$$s_3 + \frac{1}{2} s_2$$

$$= \frac{1}{2}(s_2 + s_1) + \frac{1}{2} s_2$$

$$= s_2 + \frac{1}{2} s_1$$

$$= \boxed{\frac{1}{2}(s_1 + 2s_2)}$$

$$s_4 + \frac{1}{2} s_3 = \frac{1}{2}(s_3 + s_2) + \frac{1}{2} s_3$$

$$= s_3 + \frac{1}{2} s_2$$

$$= \frac{1}{2}(s_1 + 2s_2)$$

Prob: If $\langle a_n \rangle$ be a sequence of +ve integers such that

$a_n = \sqrt{a_{n-1} \cdot a_{n-2}}$ for $n > 2$;

then show that the seq. converge to $(a_1 \cdot a_2^2)^{\frac{1}{3}}$.

HW

$$\boxed{l = \frac{1}{3}(s_1 + 2 s_2)}$$

Similarly, by considering

$s_1 > s_2$,

we will have the same limit.