Maths Optional

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$$\frac{a_{n+1}}{a_n} = \frac{\cancel{x^{n+1}}^{\,x}}{\cancel{\lfloor\underline{n+1}}_{\,(n+1)}} \times \frac{\cancel{\lfloor\underline{n}}}{\cancel{x^n}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1}$$

$$= 0 < 1$$

$\therefore \lim a_n = 0$

i.e. $\lim \dfrac{x^n}{\lfloor\underline{n}} = 0.$

Prob: Show that for any x,

$$\lim_{n\to\infty} \frac{x^n}{\lfloor\underline{n}} = 0.$$

$\longrightarrow$ Let $a_n = \dfrac{x^n}{\lfloor\underline{n}}$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\dfrac{x^{n+1}}{\lfloor\underline{n+1}}\right)}{\left(\dfrac{x^n}{\lfloor\underline{n}}\right)}$$

prob: Show that

$$\lim_{n\to\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{\lfloor n} x^n = 0$$

if $|x| < 1$.

$\rightarrow$ let $a_n = \dfrac{m(m-1)(m-2)\cdots(m-(n-1))}{\lfloor n} x^n$

$$a_{n+1} = \frac{m(m-1)\cdots(m-n)}{\lfloor n+1} x^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{m(m-1)\cdots(m-(n-1))(m-n)}{\lfloor n+1} x^{n+1}}{\dfrac{m(m-1)(m-2)\cdots(m-(n-1))}{\lfloor n} x^n}$$

$$= \frac{m-n}{n+1} \cdot x$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{\frac{m}{n}-1}{1+\frac{1}{n}}\right) x$$

$$= -x$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^K}{(1+p)^{n+1}}}{\frac{n^K}{(1+p)^n}}$$

$$= \left(\frac{n+1}{n}\right)^K \cdot \frac{1}{1+p}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^k \cdot \frac{1}{1+p}$$

$$= \frac{1}{1+p} < 1 \quad \text{for } p>0$$

$\therefore \lim a_n = 0.$

$|l| = |-x| = |x| < 1$ (Given)

$\therefore \lim a_n = 0$

<u>prob</u>: Prove that if $p>0$ then $\lim_{n\to\infty} \frac{n^K}{(1+p)^n} = 0$.

$\rightarrow$ Let $a_n = \frac{n^K}{(1+p)^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(1+y)^{n+1}}{\lfloor\underline{n+1}}}{\dfrac{(1+y)^n}{\lfloor\underline{n}}}$$

$$= \frac{1+y}{(n+1)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+y}{n+1}$$

$$= 0 < 1$$

$$\therefore \lim a_n = 0.$$

Prob: Show that

$$\lim_{n\to\infty} \frac{(1+y)^n}{\lfloor\underline{n}} = 0 \quad \forall y$$

$\rightarrow$ Let $a_n = \dfrac{(1+y)^n}{\lfloor\underline{n}}$

$$a_{n+1} = \frac{(1+y)^{n+1}}{\lfloor\underline{n+1}}$$

i.e. $a_n \in \,]l-\varepsilon, l+\varepsilon[$ for infinitely many values of n.

i.e. $|a_n - l| < \varepsilon$ for infinitely many values of n.

For any $\varepsilon > 0$,

Limit points of a seq:

A real number l is said to be a limit pt. of a seq. $\langle a_n \rangle$ if every nbd. of l contains infinitely many terms of the seq.

or

For any $\varepsilon > 0$, $]l-\varepsilon, l+\varepsilon[$ contains infinitely many terms of the seq. $\langle a_n \rangle$

Ex: $\langle 1+(-1)^n \rangle$

let $a_n = 1+(-1)^n$

$\langle 0, 2, 0, 2, 0, 2, \ldots \rangle$

0 & 2 are limit point of the seq.

for any $\varepsilon > 0$, $]0-\varepsilon, 0+\varepsilon[$

i.e. $]-\varepsilon, \varepsilon[$ —— contains infinitely many terms of the seq.

Note: ① limit point – Accumulation point, condensation point, cluster point.

② l is not a limit point of a seq. $\langle a_n \rangle$ if for some $\varepsilon > 0$

$]l-\varepsilon, l+\varepsilon[$ contains finitely many terms of the seq.

For any $\varepsilon > 0$,

$a_n \in]2-\varepsilon, 2+\varepsilon[$

for all even n.

$a_2, a_4, a_6, a_8, \ldots$

$\therefore$ 2 is a l.p. of the seq. $\langle a_n \rangle$.

$0 \in]-\varepsilon, \varepsilon[$

$a_n \in]-\varepsilon, \varepsilon[$

for all odd n.

$\therefore$ 0 is a l.p. of $\langle a_n \rangle$.

Ex: $\langle \frac{1}{n} \rangle$

Let $a_n = \frac{1}{n}$

$$\langle 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \ldots \rangle$$

Infinitely many terms of the seq. $\langle a_n \rangle$

$\boxed{0 \text{ is a l.p. of the seq.}}$

Ex: $\langle (-1)^n \rangle$

Let $a_n = (-1)^n$

$$\langle -1, 1, -1, 1, -1, 1, \ldots \rangle$$

-1 & 1 are limit points

Range set $= \{-1, 1\}$ $\rightarrow$ has no limit point.

Ex: $\langle n^2 \rangle$

$\langle 1^2, 2^2, 3^2, 4^2, \ldots \rangle$

No. limit pt.

Range set $= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

limit pt is $\underline{0}$.

infinitely many elements of the set

$\because \lim a_n = l$.

$\therefore$ corr. to $\varepsilon > 0$, $\exists$ a +ve int. m s.t.

$$|a_n - l| < \varepsilon \quad \forall\, n \geq m$$

i.e. $l - \varepsilon < a_n < l + \varepsilon$ $\forall\, n \geq m$

$a_n \in \,]l-\varepsilon, l+\varepsilon[$ for infinitely many values of n.

Th: If $\lim a_n = l$, then l is the unique limit point of $\langle a_n \rangle$. Converse is not true.

Proof: First, we show that l is a limit point of the seq. $\langle a_n \rangle$.

Let $\varepsilon > 0$ be any given number.

$$|a_n - l| < \frac{\varepsilon}{2} \quad \forall n \geq i \quad \text{--- (1)}$$

$\because$ l' is a limit point of $\langle a_n \rangle$ $\therefore$ $\exists$ a +ve int j s.t.

$$a_j \in \left] l' - \frac{\varepsilon}{2}, l' + \frac{\varepsilon}{2} \right[$$

for $j > i$ (for infinitely many values of j)

$\therefore$ l is a limit point of the seq. $\langle a_n \rangle$.

<u>uniqueness</u> Let l' be any other limit point of $\langle a_n \rangle$

<u>Claim:</u> $l = l'$.

$\because$ $\lim a_n = l$.

$\therefore$ corr. to $\frac{\varepsilon}{2} > 0$, $\exists$ a +ve int. i s.t.

Now $|l - l'| = |(l - a_j) + (a_j - l')|$

$$\leq |l - a_j| + |a_j - l'|$$

$$= |a_j - l| + |a_j - l'|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So, $|l - l'| < \varepsilon$

$\because \varepsilon$ is small, +ve and arbitrary.

i.e.

$$l' - \frac{\varepsilon}{2} < a_j < l' + \frac{\varepsilon}{2}$$

for $j > i$

i.e. $|a_j - l'| < \frac{\varepsilon}{2}$, for $j > i$ (ii)

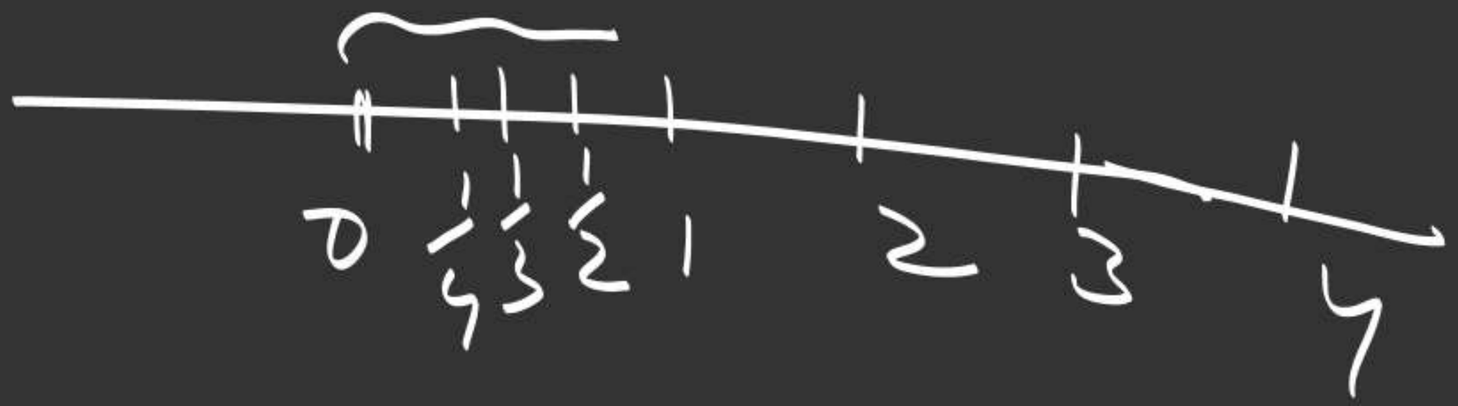
$\because j > i$

$\therefore$ (i) $\implies$

$$|a_j - l| < \frac{\varepsilon}{2}$$

Consider the example.

$$\langle 1, 2, \tfrac{1}{2}, 3, \tfrac{1}{3}, 4, \tfrac{1}{4}, \ldots \rangle$$



0 is a limit pt.

$\therefore$ it can be made smaller and smaller.

$\therefore |l - l'| = 0$

$\Rightarrow l - l' = 0$

$\Rightarrow \underline{l = l'}$.

Converse is not true

Note (1) An unbounded seq. may or may not have a limit pt.

Ex: (i) $\langle 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4} \dots \rangle$

Limit pt — '0'.

unbounded seq.

(ii) $\langle n^2 \rangle$ — has no limit point.

unbounded.

This seq. has unique limit point.

But it is not cgt.

Bolzano weierstrass th. for sequence:

Every bdd seq. has a limit point.

l is the unique limit point of the seq. $\langle a_n \rangle$.

Let $\varepsilon > 0$ be any number.

$a_n \in]l-\varepsilon, l+\varepsilon[$ — ①

for infinitely many values of n.

Claim: $\exists$ finitely many terms of the seq., say $a_{m_1}, a_{m_2}, a_{m_3}$ — a_{m_k} that

Note ② The set of limit points of a bounded seq. is bounded.

Th: Prove that a bounded seq. with a unique limit point is convergent.

Proof: Let $\langle a_n \rangle$ be a bdd. seq.

Let $\underline{m-1} = \max\{m_1, m_2, \cdots m_k\}$

① $\Rightarrow$

$$a_n \in \,]l-\varepsilon, l+\varepsilon[\quad \forall n \geq m$$

$$\Rightarrow l-\varepsilon < a_n < l+\varepsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n - l| < \varepsilon \quad \forall n \geq m$$

$\therefore a_n \to l.$

do not belong to $]l-\varepsilon, l+\varepsilon[$.

For otherwise, the infinitely many terms of the seq. not belonging to $]l-\varepsilon, l+\varepsilon[$ will have a limit point (By BWT) contrary the uniqueness of the limit point 'l'.

Ex: (11) $\langle 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots \rangle$

unique limit point — '0'.

Bounded ×

Not cgt.

Remark: Condition of the th. can not be relaxed.

Ex: (1) $\langle (-1)^n \rangle$

bdd.

unique limit pt. ×

Not cgt

Th: A necessary and sufficient condition for the convergence of a seq. is that it is bounded and has a unique limit point.