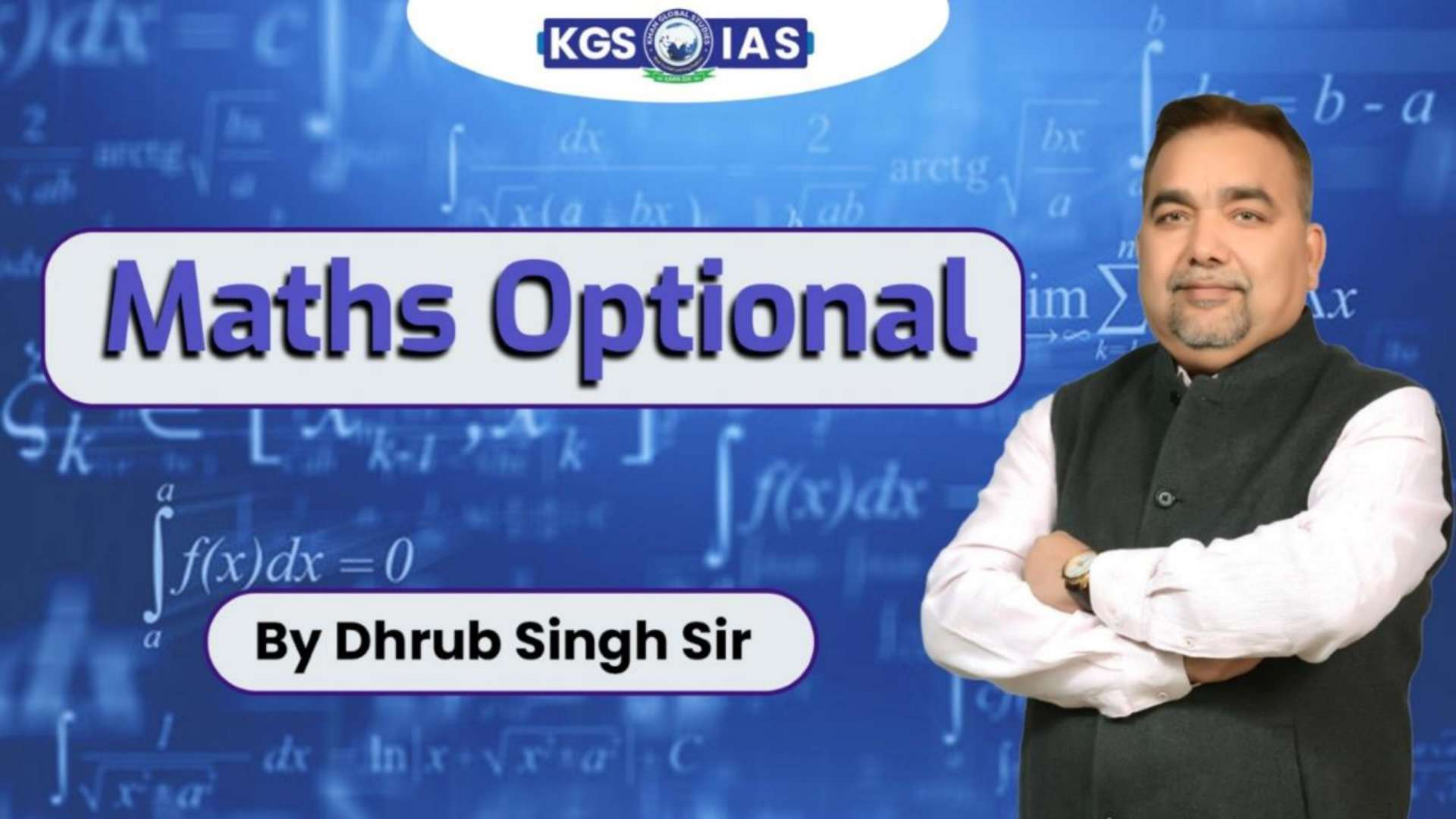
KGS IAS
Maths Optional
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$1+nh > G$

$\Rightarrow nh > G-1$

$\Rightarrow n > \frac{G-1}{h}$.

Let m be a +ve int. greater than $\frac{G-1}{h}$.

$\therefore |r^n| = r^n > G$

$\forall n \geq m$.

$\therefore \lim r^n = +\infty$.

Prob: Show that the sequence $\langle r^n \rangle$ converges iff $-1 < r \leq 1$.

$(|r| < 1)$

$r = 1$

$\rightarrow$ (1) $r > 1$

Let $r = 1+h$, $h > 0$

$\Rightarrow r^n = (1+h)^n > 1+nh$

$\forall n \in \mathbb{N}$.

Let $G > 0$ be a real no. (May be very large)

Let $\varepsilon > 0$ be given.

$$\frac{1}{1+nh} < \varepsilon$$

if $1+nh > \frac{1}{\varepsilon}$

i.e. if $nh > \frac{1}{\varepsilon} - 1$

i.e. if $n > \frac{\frac{1}{\varepsilon}-1}{h}$

Let m be a +ve int $> \frac{\frac{1}{\varepsilon}-1}{h}$.

$\therefore |x^n| < \varepsilon \quad \forall n \geq m$

$\therefore \lim x^n = 0.$

(ii) $x = 1$

$$\lim_{n\to\infty} x^n = 1$$

(iii) $-1 < x < 1$

i.e. $|x| < 1$

Let $|x| = \frac{1}{1+h}$, $h > 0$

$$|x^n| = |x|^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh} \quad \forall n \in \mathbb{N}$$

$$\langle r^n \rangle = \langle (-t)^n \rangle$$

$$= \langle (-1)^n \cdot t^n \rangle$$

oscillate infinitely.

Thus: $\langle r^n \rangle$ converges iff $-1 < r \leq 1$.

(IV) $r = -1$

$\langle (-1)^n \rangle$ — oscillate finitely.

(V) $r < -1$

Let $t = -r$

$\therefore t > 1$

$\therefore \langle (-1)^n \rangle \to l$.

$\therefore$ corr. to $\varepsilon = \frac{1}{2}$, $\exists$ a +ve int. m s.t.

$$|(-1)^n - l| < \frac{1}{2} \quad \forall n \geq m$$

i.e.

$$|-1 - l| < \frac{1}{2} \quad \forall n \geq m \quad (n \text{ is odd})$$

i.e.

$$|1 + l| < \frac{1}{2} \quad \forall n \geq m \quad (n \text{ is odd})$$

&

$$|1 - l| < \frac{1}{2} \quad \forall n \geq m \quad (n \text{ is even})$$

Note: $x^n \to 0$ iff $-1 < x < 1$.

prob: Show that $\langle (-1)^n \rangle$ does not converge.

Sol^n: Let, if possible,

$$\langle (-1)^n \rangle \longrightarrow l$$

Choose $\varepsilon = \frac{1}{2} > 0$

Th: If $\lim a_n = a$ and $a_n \geq 0 \ \forall n$, then $a \geq 0$.

proof: Let, if possible, $a < 0$

$\therefore -a > 0$

choose $\varepsilon = -\frac{1}{2} a > 0$

$\because a_n \to a$

$\therefore$ corr. to $\varepsilon = -\frac{1}{2} a$, $\exists$ a +ve int. m s.t.

$$2 = |1+\ell + 1-\ell|$$

$$\leq |1+\ell| + |1-\ell|$$

$$< \frac{1}{2} + \frac{1}{2}$$

i.e. $2 < 1$

a contradiction.

$\therefore$ our supposition is wrong.

$\therefore \langle (-1)^n \rangle$ is not cgt.

which is a contradiction to the fact that

$a_n \geq 0 \quad \forall n.$

$\therefore$ our supposition is wrong.

$\therefore a \geq 0$

$$|a_n - a| < \varepsilon \quad \forall n \geq m.$$

$$\Rightarrow a - \varepsilon < a_n < a + \varepsilon, \quad \forall n \geq m$$

$$\Rightarrow a + \tfrac{1}{2}a < a_n < a - \tfrac{1}{2}a$$

$$\Rightarrow \tfrac{3}{2}a < a_n < \tfrac{1}{2}a \quad \forall n \geq m$$

i.e. $a_n < \frac{1}{2}a \quad \forall n \geq m$

i.e. $a_n < 0 \quad \forall n \geq m$

Th: If $\langle a_n \rangle$, $\langle b_n \rangle$ are two sequences s.t

(I) $a_n \leq b_n \ \forall n$.

(II) $\lim a_n = a$ and $\lim b_n = b$

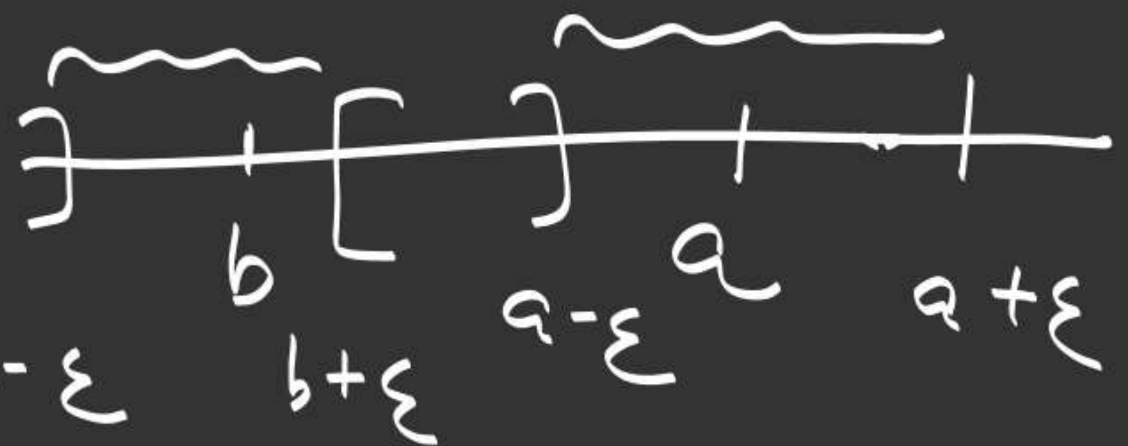
then $a \leq b$.

Proof: Let, if possible.

$$a > b$$

$$\therefore a - b > 0$$

choose $\varepsilon = \dfrac{a-b}{3}$



obviously,

$$]b-\varepsilon, b+\varepsilon[\, \cap \,]a-\varepsilon, a+\varepsilon[\; = \phi \quad \text{—(A)}$$

$\because \lim a_n = a$ and $\lim b_n = b$

$\therefore$ corr. to the given $\varepsilon > 0$

$\therefore b_n < a_n \quad \forall n \geq m.$

using (A)

which is a contradiction to the fact that $a_n \leq b_n$ $\forall n.$

$\therefore$ our supposition is wrong

$\therefore a \leq b.$

$\exists$ +ve int. m_1 & m_2

s.t.

$|a_n - a| < \varepsilon, \ \forall n \geq m_1$ — (I)

$|b_n - b| < \varepsilon, \ \forall n \geq m_2$ — (II)

let $m = \max\{m_1, m_2\}$

$\therefore$ (I) & (II) hold

$\forall n \geq m.$

$\therefore$ corr. to $\varepsilon > 0$, $\exists$ +ve integers m_1 & m_2 s.t.

$$|a_n - l| < \varepsilon \quad \forall n \geq m_1$$

i.e. $l - \varepsilon < a_n < l + \varepsilon \quad \forall n \geq m_1$ — (I)

& $|c_n - l| < \varepsilon \quad \forall n \geq m_2$

i.e. $l - \varepsilon < c_n < l + \varepsilon \quad \forall n \geq m_2$ — (II)

Let $m = \max\{m_1, m_2\}$

$\therefore$ (I) and (II) hold $\forall n \geq m$

Sandwich th:

Stat: If $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ are three sequences s.t.

(I) $a_n \leq b_n \leq c_n \quad \forall n$.

(II) $\lim a_n = \lim c_n = l$

then $\lim b_n = l$.

Proof: Let $\varepsilon > 0$ be given

$\therefore \lim a_n = l$

& $\lim c_n = l$.

Prob: Show that the seq. $\langle b_n \rangle$, where

$$b_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$$

Converges to zero.

$\rightarrow$ $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$

$\leq \frac{1}{n^2} + \frac{1}{n^2} + \cdots$ n times $= n \cdot \frac{1}{n^2} = \frac{1}{n}$

$\therefore b_n \leq \frac{1}{n} \quad \forall n$ — (i)

Given

$a_n \leq b_n \leq c_n \quad \forall n$ — (iii)

By (i), (ii) & (iii)

$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon$

$\forall n \geq m$

$\Rightarrow l - \varepsilon < b_n < l + \varepsilon \quad \forall n \geq m$

i.e. $|b_n - l| < \varepsilon \quad \forall n \geq m$.

i.e. $\lim b_n = l$

$\therefore \lim \frac{1}{4n} = 0$

$\& \lim \frac{1}{n} = 0$

$\therefore$ By sandwich th,

$\lim b_n = 0.$

$$\frac{1}{(n+n)^2} + \frac{1}{(n+n)^2} + \cdots \underline{\quad}_{n \text{ times}} \leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$$

i.e. $\frac{n}{(2n)^2} \leq b_n \quad \forall n$

i.e. $\frac{1}{4n} \leq b_n \quad \forall n$ —— (ii)

from (i) & (ii)

$$\frac{1}{4n} \leq b_n \leq \frac{1}{n} \quad \forall n.$$

$$\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \cdots n \text{ times}$$

$$\frac{n}{\sqrt{n^2+n}} \leq b_n \quad \forall n \text{ —— (ii)}$$

(i) and (ii) $\implies$

$$\frac{n}{\sqrt{n^2+n}} \leq b_n \leq \frac{n}{\sqrt{n^2+1}} \quad \forall n.$$

$$\lim \frac{n}{\sqrt{n^2+n}} = \lim \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n}}} = 1$$

$$\lim \frac{n}{\sqrt{n^2+1}} = \lim \frac{\cancel{n}}{\cancel{n}\cdot\sqrt{1+\frac{1}{n^2}}} = 1$$

<u>Prob</u>: Show that the seq. $\langle b_n \rangle$ where

$$b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Converges to 1.

$\rightarrow$

$$b_n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots n \text{ times}$$

$$b_n \leq \frac{n}{\sqrt{n^2+1}} \quad \forall n \text{ —— (i)}$$

$b_n = a_n - l.$

$$\lim b_n = \lim(a_n - l)$$
$$= \lim a_n - l.$$
$$= l - l = 0$$

Now

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$= \frac{b_1 + l + b_2 + l + \cdots + b_n + l}{n}$$

$$= \left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) + l$$

$\therefore$ By Sandwich th.

$$\lim b_n = 1$$

Cauchy's first th. on limits

Stat: If $\lim a_n = l$ then

$$\lim\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) = l.$$

Proof: Let $b_n = a_n - l$

$\therefore a_n = b_n + l$ then.

Let $\varepsilon > 0$ be given.

$\because$ $b_n \to 0$

$\therefore$ corr. to $\frac{\varepsilon}{2} > 0$, $\exists$ a +ve int m s.t.

$$|b_n - 0| < \frac{\varepsilon}{2} \quad \forall n \geq m$$

i.e. $|b_n| < \frac{\varepsilon}{2} \quad \forall n \geq m$ — (ii)

Now $\left|\frac{b_1 + b_2 + \cdots\cdots + b_n}{n}\right|$

Here we have to prove equivalently,

$$\lim \frac{b_1 + b_2 + \cdots + b_n}{n} = 0$$

$\because$ $\langle b_n \rangle$ is CgA.

$\therefore$ It is bdd.

$\therefore$ $\exists$ a +ve real no K s.t.

$$|b_n| \leq K \quad \forall n \text{ — (i)}$$

$$< \frac{mK}{n} + \frac{\varepsilon}{2} \quad \forall n \geq m, \quad \text{(iii)}$$

$$\frac{mK}{n} < \frac{\varepsilon}{2} \text{ if } \frac{n}{mK} > \frac{2}{\varepsilon}$$

i.e. if $n > \dfrac{2mK}{\varepsilon}$

let m_1 ~~be~~ a +ve int. $> \dfrac{2mK}{\varepsilon}$

$$\therefore \frac{mK}{n} < \frac{\varepsilon}{2} \quad \forall n \geq m_1 \quad \text{(iv)}$$

$$= \left| \frac{(b_1 + b_2 + \cdots + b_m)}{n} + \frac{(b_{m+1} + \cdots + b_n)}{n} \right|$$

$$\leq \frac{|b_1 + b_2 + \cdots + b_m|}{n} + \frac{|b_{m+1} + \cdots + b_n|}{n}$$

$$\leq \frac{|b_1| + |b_2| + \cdots + |b_m|}{n} + \frac{|b_{m+1}| + |b_{m+2}| + \cdots + |b_n|}{n}$$

$$< \frac{mK}{n} + \left(\frac{n-m}{n}\right) \cdot \frac{\varepsilon}{2} \quad \forall n \geq m$$

using (i) & (ii)

<u>Prob:</u> Prove that

$$\lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right]=0$$

<u>Sol^n:</u> Let $a_n = 1/n$

$$\lim a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

By Cauchy's 1st th. on limits

$$\lim_{n\to\infty} \frac{1}{n}(a_1+a_2+\cdots+a_n)=0$$

Let $m_0 = \max\{m, m_1\}$

$\therefore$ (III) & (IV) $\Rightarrow$

$$\left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

$\forall n \geq m_0$

i.e. $\lim \dfrac{b_1+b_2+\cdots+b_n}{n} = 0$

$$\frac{a_1+a_2+\cdots+a_n}{n} = \begin{cases} 0, & n \text{ is even} \\ -\frac{1}{n}, & n \text{ is odd} \end{cases}$$

$$\therefore \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 0$$

But $\langle a_n \rangle$ is not cgt.

i.e. $\lim \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = 0$

<u>Note</u>: Converse of Cauchy's 1st th. on limits is not true.

Consider the example.

$\langle (-1)^n \rangle$,

let $a_n = (-1)^n$

$$\lim a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}}$$

$$= \frac{1}{\sqrt{1+0}} = 1$$

$\therefore$ By Cauchy's 1st th. on limit,

Prob: Show that

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right] = 1$$

$\longrightarrow$ i.e. to prove

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \cdots + \frac{n}{\sqrt{n^2+n}}\right] = 1$$

Let $a_n = \dfrac{n}{\sqrt{n^2+n}}$

$$\lim_{n\to\infty} \frac{1}{n}\left[a_1 + a_2 + \cdots + a_n\right] = 1$$

$$\text{i.e. } \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \text{—} + \frac{n}{\sqrt{n^2+n}}\right] = 1$$

$$\text{i.e. } \lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right] = 1$$