KGS IAS

Maths Optional

By Dhrub Singh Sir

→ let $x = n$ — an int.

$$\lim_{x\to n^-} f(x) = \lim_{\substack{x\to n \\ x<n}} |x| + [x]$$

$$= |n| + n - 1$$

$$\lim_{x\to n^+} f(x) = \lim_{\substack{x\to n \\ x>n}} |x| + [x]$$

$$= |n| + n$$

$$f(n) = |n| + n.$$

Prob: Discuss the continuity of $f(x) = |x| + [x]$

→

$$f(x) = \begin{cases} -x-2 & -2 \le x < -1 \\ -x-1 & -1 \le x < 0 \\ x+0 & 0 \le x < 1 \\ x+1 & 1 \le x < 2 \\ x+2 & 2 \le x < 3 \end{cases}$$

$f(\alpha) = |\alpha| + n$

$\therefore \lim\limits_{x \to \alpha} f(x) = f(\alpha)$

$\therefore f(x)$ is cts at $\underline{x = \alpha}$.

$\because$ L.H.L $\neq$ R.H.L.

$\therefore f$ is disc. at $\underline{x = n}$

i.e., f is disc. at integral point.

Let $\alpha \in \mathbb{R} \sim \mathbb{Z}$.

$\therefore \exists$ some integer n s.t.

$$n < \alpha < n+1$$

$$\lim_{x \to \alpha} f(x) = \lim_{x \to \alpha} |x| + [x] = |\alpha| + n$$

$\lim_{x \to 0} \sin\frac{1}{x}$ doesn't exist.

$\therefore$ $f(x)$ is discontinuous at

$x = 0, 2$.

Prob: Examine the continuity of the function

$$f(x) = \begin{cases} 2x - [x] + \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0, 2$

$\longrightarrow$ $\lim_{x \to 0} [x]$ doesn't exist.

$\lim_{x \to 2} [x]$ ".

$$\lim_{x\to n+} f(x) = \lim_{\substack{x\to n\\ x>n}} x-[x]-\frac{1}{2}$$

$$= \cancel{x}-\cancel{x}-\frac{1}{2}$$

$$= -\frac{1}{2}$$

$$f(n) = 0$$

$\therefore \lim_{x\to n} f(x)$ does not exist.

$\therefore$ f is disc. at $x = \underline{n}$.

prob: Let f be the function defined on R by setting

$$f(x) = \begin{cases} x-[x]-\frac{1}{2}, & x \text{ is not an int.} \\ 0, & x \text{ is an int.} \end{cases}$$

Discuss the continuity.

$\rightarrow$ Let $x=n$ be an integer.

$$\lim_{x\to n-} f(x) = \lim_{\substack{x\to n\\ x<n}} x-[x]-\frac{1}{2} = \cancel{n}-(\cancel{n}-1)-\frac{1}{2} = \frac{1}{2}.$$

$\therefore \lim_{x \to \alpha} f(x) = f(\alpha)$

$\therefore$ f is cts at $x = \alpha$.

Let $\alpha \in \mathbb{R} \sim \mathbb{Z}$.

$\therefore \exists$ some integer n, s.t

$n < \alpha < n+1$

$$\lim_{x \to \alpha} f(x) = \lim_{x \to \alpha} x - [x] - \frac{1}{2}$$

$$= \alpha - n - \frac{1}{2}$$

$$f(\alpha) = \alpha - n - \frac{1}{2}$$

$\rightarrow$

$$f(x) = \begin{cases} 1, & \frac{1}{2} < x \le 1 \\ \frac{1}{2}, & \frac{1}{2^2} < x \le \frac{1}{2} \\ \frac{1}{2^2}, & \frac{1}{2^3} < x \le \frac{1}{2^2} \\ \vdots & \vdots \\ \vdots & \vdots \end{cases}$$

<u>At $x=0$</u>

$$\lim_{x\to 0+} f(x) = \lim_{\substack{x\to 0 \\ x>0}} f(x)$$

$$= \lim_{n\to\infty} \frac{1}{2^n} = 0 = f(0)$$

$\therefore$ f is cts at $x=0$.

<u>Prob</u>: Show that the function f defined on $[0,1]$ as:

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \\ & (n = 0, 1, 2, 3, \ldots) \\ 0, & \text{when } x = 0 \end{cases}$$

is cts except at the points $x = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$. Describe the nature of discontinuity.

At $x = \frac{1}{2}$

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ x < \frac{1}{2}}} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ x > \frac{1}{2}}} 1 = 1$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\therefore \lim_{x \to \frac{1}{2}^-} f(x) \neq \lim_{x \to \frac{1}{2}^+} f(x)$$

At $x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{\substack{x \to 1 \\ x < 1}} 1 = f(1)$$

$\therefore$ f is ct at $x = 1$.

f being constant is ct $\forall x \in]0, 1[$ except possibly at $x = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

Prob: Let f be the function defined on $[0,1]$ as

$$f(x) = \begin{cases} (-1)^r & \text{if } \frac{1}{r+1} \le x < \frac{1}{r}, \quad (r = 1, 2, 3, \ldots) \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1. \end{cases}$$

Examine the continuity of f at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

f is discts at $x = \frac{1}{2}$.

Disc. of 1st kind.

Similiarly, we can show that f is discts at $x = \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$

At $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{\substack{x\to 1\\ x<1}} (-1) = -1$$

$$f(1) = 1$$

$\therefore$ f is disc. at $x=1$.

At $x=\frac{1}{2}$

$$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{\substack{x\to \frac{1}{2}\\ x<\frac{1}{2}}} 1 = 1$$

$$f(x) = \begin{cases} -1, & \frac{1}{2} \le x < 1 \\ 1, & \frac{1}{3} \le x < \frac{1}{2} \\ -1, & \frac{1}{4} \le x < \frac{1}{3} \\ \vdots & \vdots \\ 1, & x = 1 \end{cases}$$

Similarly, we can show that

f is disc. at $x = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

<u>prob</u>: Let f be a function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational.} \end{cases}$$

Show that f is discont. at every point in $\mathbb{R}$.

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ x > \frac{1}{2}}} (-1) = -1$$

$$f\left(\tfrac{1}{2}\right) = -1$$

$\because \lim_{x \to \frac{1}{2}^-} f(x) \neq \lim_{x \to \frac{1}{2}^+} f(x)$

$\therefore$ f is disc. at $x = \frac{1}{2}$

As $n \to \infty$,

$|a_n - a| \to 0$

i.e. $a_n \to a$

$\because$ Each a_n is irrational.

$\therefore f(a_n) = -1 \quad \forall n \in \mathbb{N}$

$\therefore \langle f(a_n) \rangle \to -1 \neq f(a)$

$\therefore f$ is discontinuous at $x = a$.

Solⁿ: Case ① Let a be a rational no.

$f(a) = 1$

For each $n \in \mathbb{N}$, choose an irrational number

$a_n \in \left] a - \frac{1}{n}, a + \frac{1}{n} \right[$

$\therefore a - \frac{1}{n} < a_n < a + \frac{1}{n} \quad \forall n \in \mathbb{N}$

$\Leftrightarrow |a_n - a| < \frac{1}{n}$

$\therefore \langle f(b_n) \rangle \to 1 \neq f(b)$

$\therefore$ f is discts at $x = b$.

Thus f is discts $\forall$ points in $\mathbb{R}$.

Case (ii) Let b be an irrational number.

$$\therefore f(b) = -1$$

Also we can have a seq. of rational number $\langle b_n \rangle \to b$.

For each $n \in \mathbb{N}$.

$$f(b_n) = 1$$

Prob: Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is cts only at $x = 0$.

$\longrightarrow$

Case ① Let a be an irrational no.

$$\therefore f(a) = a.$$

Now we can construct a seq.

(HW) Prob: Show that the function f defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discts at every point in $\mathbb{R}$

(Dirichlet's fn.)

$\therefore$ f is discts at $x = \underline{a}$.

<u>Case (II)</u> Let $b \neq 0$ be a rational

no.

$\therefore f(b) = -b$

Now we can construct a seq. of irrational number $\langle b_n \rangle$ converging to 'b'.

$\therefore$ For each $n \in \mathbb{N}$

$f(b_n) = b_n$

of ration no. $\langle a_n \rangle \to a$

$\therefore$ For each $n \in \mathbb{N}$.

$f(a_n) = -a_n$

Now

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (-a_n)$$

$$= -1 \times \lim_{n\to\infty} a_n$$

$$= -a \neq f(a)$$

$\therefore \langle f(a_n) \rangle \not\to f(a)$

$|f(x)-f(0)| = |f(x)|$

$= |-x|$, when x irrational.

$= |x|$

$\therefore |f(x)-f(0)| < \varepsilon$ when $|x-0| < \varepsilon$

choose $\delta = \varepsilon$

$\therefore |x-0| < \delta \Rightarrow |f(x)-f(0)| < \varepsilon$
(x irrational)

Also $|f(x)-f(0)| = |f(x)|$
$= |x|$

$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} b_n$

$= b \neq f(b)$

$\therefore \langle f(b_n) \rangle \not\to f(b)$

So, f is discts at $x = b$

Case (iii) At $x = 0$.

let $\varepsilon > 0$ be given

HW

Prob: Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ irrational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is cts only at $x = 0$.

$\therefore |f(x) - f(0)| < \varepsilon$ when $|x| < \varepsilon$

By choose $\delta = \varepsilon$

$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$

(when x is irr.)

Then

$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$

$x \in \mathbb{R}$

$\therefore f$ is cts at $x = 0$

Case (ii) b is an irrational no.

(prove it!)

Case (iii) $x = 3$

Let $\varepsilon > 0$ be given

when x is rational

$$|f(x) - f(3)| = |2x - 6|$$
$$= 2|x-3|$$

$|f(x) - f(3)| < \varepsilon$ if $|x-3| < \frac{\varepsilon}{2}$

Let $\delta_1 = \frac{\varepsilon}{2}$

Prob: Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2x, & \text{if } x \text{ is rational} \\ x+3, & \text{if } x \text{ is irrational.} \end{cases}$$

Examine the continuity of $g(x)$.

→ Case (i) a is a rational no. ($a \neq 3$)

(prove it!)

Let $\delta = \min\{\delta_1, \delta_2\}$.

$\therefore$ (I) & (II) hold for this choice of δ.

So $|f(x) - f(3)| < \varepsilon$ when

$|x - 3| < \delta$

$x \in \mathbb{R}$

$\therefore$ f is cts at $x = 3$

$\therefore |x - 3| < \delta_1 \Rightarrow |f(x) - f(3)| < \varepsilon$ — (I)

When x is irrational

$$|f(x) - f(3)| = |x + 3 - 6|$$
$$= |x - 3|$$

So, $|f(x) - f(3)| < \varepsilon$ if $|x - 3| < \varepsilon$

Let $\delta_2 = \varepsilon$

$\therefore |x - 3| < \delta_2 \Rightarrow |f(x) - f(3)| < \varepsilon$ — (II)

$$f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 1, & x > 1 \end{cases}$$

<u>At $x = 0$</u>

$$\lim_{x\to 0+} f(x) = 0 = f(0)$$

$\therefore$ f is cts at $x = 0$

<u>At $x = 1$</u> $\quad \lim_{x\to 1-} f(x) = 0 \qquad f(1) = \frac{1}{2}$

$$\lim_{x\to 1+} f(x) = 1$$

<u>Prob:</u> Let f be the function defined on $[0, \infty[$ by setting

$$f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}, \quad \forall x \geq 0$$

Examine the continuity of f in $[0, \infty[$

$\rightarrow$

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$$

$f(x)$ is disc. at $x=1$

$\rightarrow$ For $0<x<1$, $x>1$

$f(x)$ is cts.

$\therefore$ is cts.

so f is cts in $[0,\infty[$

except $x=1$.