KGS IAS

Maths Optional

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$\sum \frac{1}{2^n}$ being a geometric series and having common ratio $\frac{1}{2}$ (<1) is cgt.

$\therefore$ By comparison test,

$\sum \frac{1}{n^n}$ is cgt.

Prob: Test for convergence of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$$

$\longrightarrow$ $n > 2$

$\Rightarrow n^n > 2^n$

$\Rightarrow \frac{1}{n^n} < \frac{1}{2^n} \quad \forall n > 2$

$\Rightarrow e^{n^2} > n^2$

$\Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2} \quad \forall n.$

$\sum \frac{1}{n^2}$ is cgt.

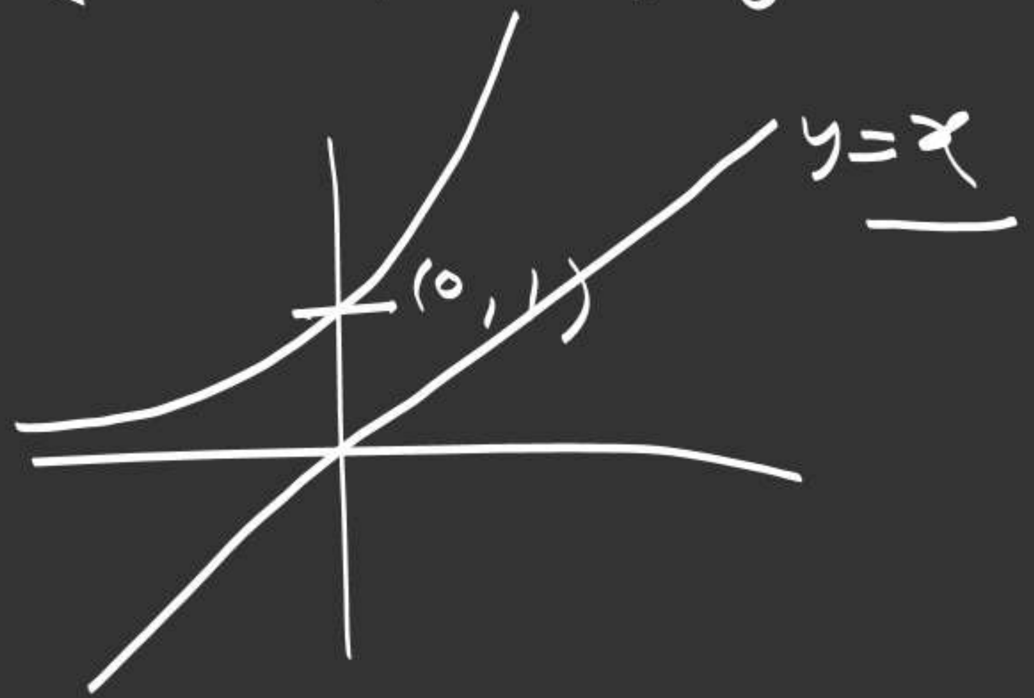
$\therefore$ By comparison test.

$\sum \frac{1}{e^{n^2}}$ is cgt.

i.e. $\sum e^{-n^2}$ is cgt.

Prob: Show that the series $\sum e^{-n^2}$ converges.

$\longrightarrow e^x > x \quad$ for $x > 0$



$\sum \frac{1}{n^2 \cdot \log n}$ is also cgt.

<u>Prob:</u> Test for convergence of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots\cdots$$

$\rightarrow$ $u_n = \frac{2n-1}{n\cdot(n+1)(n+2)} \sim \frac{2\cancel{n}}{\cancel{n}\cdot n\cdot n}$

<u>Prob:</u> Test for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \cdot \log n}.$$

$$\frac{1}{n^2\cdot \log n} < \frac{1}{n^2} \quad \forall\, n \geq 3$$

$\sum \frac{1}{n^2}$ is cgt.

$\therefore$ By comparison test

$$= \frac{2}{1\cdot 1} = 2 \neq 0.$$

$\therefore$ By limit form of comparison test,

$\sum u_n$ and $\sum v_n$ will have same nature.

$\because \sum v_n$ Converge.

$\therefore \sum u_n$ Converge.

Consider $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}}$$

$$= \frac{(2n-1)\, n^2}{n(n+1)(n+2)}$$

$$\lim \frac{u_n}{v_n} = \lim \frac{\cancel{n^2}\left(2-\frac{1}{n}\right)\cdot \cancel{n}}{\cancel{n\cdot n}\left(1+\frac{1}{n}\right)\cdot \cancel{n}\cdot\left(1+\frac{2}{n}\right)}$$

$\rightarrow u_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n+2}} \sim \dfrac{1}{\sqrt{n}+\sqrt{n}}$

$= \dfrac{1}{2\sqrt{n}}$

Let $v_n = \dfrac{1}{\sqrt{n}}$.

$$\frac{u_n}{v_n} = \frac{\dfrac{1}{\sqrt{n+1}+\sqrt{n+2}}}{\dfrac{1}{\sqrt{n}}}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$$

HW **Prob**: Test for convergence of the series

$$\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \cdots$$

Ans: convergent

Prob: Test for convergence of the series

$$\frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots$$

$\therefore \sum v_n$ diverges.

$\therefore \sum u_n$ also diverges.

Prob: Test for convergence of the series

$$\sum \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$$

→ Let $u_n = \dfrac{\sqrt{n+1} - \sqrt{n-1}}{n}$

$$\lim \frac{u_n}{v_n} = \lim \frac{1}{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$$

$$\neq 0$$

By limit form of comp. test $\sum u_n$ and $\sum v_n$ will have same nature.

let $v_n = \frac{1}{n^{3/2}}$

$$\frac{u_n}{v_n} = \frac{\dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}}{\dfrac{1}{n^{3/2}}}$$

$$= \frac{2\,\cancel{n^{3/2}}}{\cancel{\sqrt{n}\,n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

$$\lim \frac{u_n}{v_n} = \frac{2}{2} = 1 \neq 0$$

$$u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{(\cancel{n}+1)-(\cancel{n}-1)}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$$

$$= \frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)} \approx \frac{\cancel{2}}{\cancel{2}\cdot n\cdot\sqrt{n}}$$

$$= \frac{1}{n^{3/2}}$$

Prob: Test for the convergence of the series

$$\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+ \cdots$$

HW

$\rightarrow$ $u_n = \dfrac{\sqrt{n+1}-1}{(n+2)^3-1} \sim \dfrac{\sqrt{n}}{n^3} = \dfrac{1}{n^{5/2}}$

$v_n = \dfrac{1}{n^{5/2}}$ (Convergent)

$\therefore$ By limit form of comparison test.

$\sum u_n$ and $\sum v_n$ will have same nature.

$\because \sum v_n$ converge.

$\therefore \sum u_n$ also converge.

(iii) Ex: $\sum \frac{1}{n}$ $\sum \frac{1}{n^2}$

$\lim \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim \frac{1}{n^{\frac{1}{n}}} = 1$

$\sum \frac{1}{n}$ — dgt

$\lim \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim \frac{1}{\left(n^{\frac{1}{n}}\right)^2}$

$= 1.$

$\sum \frac{1}{n^2}$ converge.

Cauchy's root test: If $\sum u_n$ is a +ve term series, such that $\lim (u_n)^{\frac{1}{n}} = l$, then the series

(i) converge if $l < 1$

(ii) diverge if $l > 1$

(iii) test fails if $l = 1$.

$$\lim (u_n)^{\frac{1}{n}} = \lim \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1$$

$\therefore$ The given series converges.

(By Cauchy's root test).

Prob: Test for convergence of the series whose general term is $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

$\rightarrow$ Let $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$(u_n)^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$\Rightarrow (u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$\lim (u_n)^{\frac{1}{n}} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root test the given series converges.

Prob: Test for convergence of the series whose n^{th} term is $\dfrac{n^{n^2}}{(n+1)^{n^2}}$

$\rightarrow$ Let $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}}$

$$(u_n)^{\frac{1}{n}} = \frac{n^n}{(n+1)^n}$$

$\therefore$ By Cauchy's root test the given series converges.

Prob: Test for convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Prob: Show that the series $\sum\left(n^{\frac{1}{n}}-1\right)^n$ converges.

$\rightarrow$ Let $u_n=\left(n^{\frac{1}{n}}-1\right)^n$

$$(u_n)^{\frac{1}{n}}=n^{\frac{1}{n}}-1$$

$$\lim (u_n)^{\frac{1}{n}}=\lim\left(n^{\frac{1}{n}}-1\right)$$

$$=1-1=0<1$$

$$= \frac{1}{e-1} < 1 \qquad \because e = 2.71$$

$$e - 1 = 1.71$$

$\therefore$ By Cauchy's root test the given series converges.

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$u_n^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$\lim u_n^{\frac{1}{n}} = \lim \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \lim \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1}$$

$$= 1 \cdot \left[e - 1\right]^{-1}$$

$$\lim (u_n)^{1/n} = \lim \left[\frac{1}{2} \cdot \frac{1}{2^{\frac{(-1)^n}{n}}} \right]$$

$$= \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} < 1$$

$\therefore$ By Cauchy's root test the given series converges.

Prob: Test for convergence of the series whose n^{th} term is $2^{-n-(-1)^n}$.

$\rightarrow$ Let $u_n = \dfrac{1}{2^{n+(-1)^n}}$

$$(u_n)^{\frac{1}{n}} = \left[\frac{1}{2^n \cdot 2^{(-1)^n}} \right]^{\frac{1}{n}}$$

Remark: Cauchy's root test is stronger than Ratio test.

D'Alembert's ratio test: Let $\sum u_n$ be a +ve term series such that $\lim \frac{u_n}{u_{n+1}} = l$ Then

(I) $\sum u_n$ converges if $l > 1$

(II) $\sum u_n$ diverges if $l < 1$.

(III) Test fails when $l = 1$.

$$\frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1} \times \frac{(n+1)^2+1}{(n+1)^2-1} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\cancel{n^2}\left(1-\frac{1}{n^2}\right)}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} \times \frac{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}\right]} \cdot \frac{1}{x}$$

$$= \frac{1}{1} \times \frac{1}{1} \cdot \frac{1}{x} = \frac{1}{x}$$

By Ratio test, the series

Prob: Test for convergence of the series $\sum \frac{n^2-1}{n^2+1} x^n$, $x > 0$.

→ Let $u_n = \frac{n^2-1}{n^2+1} x^n$

$$u_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}$$

$\therefore \sum u_n$ diverges.

$\sum u_n$ converges if $x < 1$

$\sum u_n$ diverges if $x \geq 1$.

Converges if $\frac{1}{x} > 1$ i.e. $x < 1$

diverges if $\frac{1}{x} < 1$ i.e. $x > 1$

test fails when $x = 1$

For $x = 1$

$$\lim u_n = \lim \frac{n^2 - 1}{n^2 + 1}$$

$$= \lim \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}} = 1 \neq 0$$

Prob: Test for the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\cdot\frac{2}{5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots$$

$$\rightarrow \quad u_n = \left[\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdots(2n+1)}\right]^2$$

$$u_{n+1} = \left[\frac{1\cdot 2\cdot 3 \cdots n\cdot(n+1)}{3\cdot 5\cdots(2n+1)(2n+3)}\right]^2$$

$$\frac{u_n}{u_{n+1}} = \left[\frac{\cancel{1\cdot 2\cdot 3\cdots n}}{\cancel{3\cdot 5\cdots(2n+1)}} \times \frac{\cancel{3\cdot 5\cdots(2n+1)}(2n+3)}{\cancel{1\cdot 2\cdots n}\cdot(n+1)}\right]^2$$

$$\lim \frac{u_n}{u_{n+1}} = \lim\left[\frac{2n+3}{n+1}\right]^2$$

$$= \lim\left[\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right]^2 = 4 > 1$$

$\therefore$ By ratio test, the given series converges.

HW

Prob: Test for convergence of the following series

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\lfloor n}{n^n}$.

(ii) $\dfrac{x}{2\sqrt{3}} + \dfrac{x^2}{3\sqrt{4}} + \dfrac{x^3}{4\sqrt{5}} + \cdots$