KGS IAS

Maths Optional

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By Limit form of comparison test.

$\sum u_n$ diverges

as $\sum v_n$ diverges.

HW 2 cgt.

HW 3 cgt for $x \le 1$

div. for $x > 1$.

HW 1 Divergent

$$u_n = \frac{1}{\sqrt{n(n+1)}} \sim \frac{1}{n}$$

Let $v_n = \frac{1}{n}$

$$\lim \frac{u_n}{v_n} = \lim \frac{n}{\sqrt{n(n+1)}}$$

$$= \lim \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}} = 1 \neq 0$$

Prob: Test for convergence of the series

$$\frac{x}{\underline{|1}} + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots \qquad , x > 0$$

$$\rightarrow \quad u_n = \frac{x^{2n-1}}{\underline{|2n-1}}$$

$$u_{n+1} = \frac{x^{2n+1}}{\underline{|2n+1}}$$

HW (2)

$$u_n = \frac{\underline{|n}}{n^n}$$

$$u_{n+1} = \frac{\underline{|n+1}}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{\underline{|n}}}{n^n} \cdot \frac{(n+1)^{n+\cancel{1}}}{\cancel{\underline{|n+1}}\ \cancel{n+1}}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right)^n = e > 1$$

By Ratio test, cgt.

Prob: Test for convergence of the series

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots \qquad x>0$$

→ Ignoring 1st term,

$$u_n = \frac{x^{2n}}{2n}$$

$$u_{n+1} = \frac{x^{2n+2}}{2n+2}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x}^{2n-1}}{\cancel{\lfloor 2n-1}} \times \frac{\cancel{\lfloor 2n+1}}{\cancel{x^{2n+1}}} \quad 2n \times (2n+1)$$

$$x^2$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{2n \times (2n+1)}{x^2}$$

$$= \infty > 1$$

∴ By Ratio test, the series converges.

But test fails for $x=1$.

$$u_n = \frac{1}{2n} \sim \frac{1}{n}$$

Let $v_n = \frac{1}{n}$

$$\lim \frac{u_n}{v_n} = \lim \frac{1}{2}$$

$$= \frac{1}{2} \neq 0$$

By Limit form of comparison test,

$\sum u_n$ div. as $\sum \frac{1}{n}$ div.

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x^{2n}}}{2n} \cdot \frac{2n+2}{\cancel{x^{2n+2}}\, x^2}$$

$$\Rightarrow \lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right) \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

By Ratio test, the series

Conv. for $\frac{1}{x^2} > 1$ i.e. $x^2 < 1$

i.e. $x < 1$

div. for $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$

$\Rightarrow x > 1$

$$u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{x^n}{x^{n+1}} \cdot \frac{2^{n+2}+1}{2^{n+2}-2}$$

$$= \frac{2^{n+1}-2}{2^{n+2}-2} \times \frac{2^{n+2}+1}{2^{n+1}+1} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\frac{1}{2}-\frac{1}{2^{n+1}}}{1-\frac{1}{2^{n+1}}} \times \frac{1+\frac{1}{2^{n+2}}}{\frac{1}{2}+\frac{1}{2^{n+2}}} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

Prob: Test for convergence of the series

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots$$

→ Ignoring 1st term.

$$u_n = \frac{2^{n+1}-2}{2^{n+1}+1} x^n$$

$\lim u_n = \lim \dfrac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^{n+1}}}$

$= 1 \neq 0.$

$\therefore \sum u_n$ diverges for

$\underline{x = 1}$.

By Ratio test

$\sum u_n$ converges for $\frac{1}{x} > 1$

i.e. $x < 1$

" " div. " $\frac{1}{x} < 1$

i.e. $x > 1$

Test fails for $x = 1$

For $x = 1$,

$$u_n = \frac{2^{n+1} - 2}{2^{n+1} + 1}$$

than Ratio test.

prob: Test for convergence of the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \cdot \frac{1}{n}$$

$\rightarrow$ $$u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \cdot \frac{1}{n}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots 2n\cdot(2n+2)} \cdot \frac{1}{n+1}$$

Raabe's test: Let $\sum u_n$ be a +ve term series such that $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ Then

(i) $\sum u_n$ converges if $l>1$.

(ii) $\sum u_n$ diverges if $l<1$.

Test fails if $l=1$

Note: Raabe's test is stronger

Applying Raabe's test

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)$$

$$=\lim n\left[\frac{(2n+2)(n+1)}{(2n+1)\cdot n}-1\right]$$

$$=\lim \cancel{n}\left[\frac{2\cancel{n^2}+2n+2n+2-2\cancel{n^2}-n}{(2n+1)\cdot\cancel{n}}\right]$$

$$=\lim\left[\frac{3n+2}{2n+1}\right]=\lim\left[\frac{3+\frac{2}{n}}{2+\frac{1}{n}}\right]$$

$$=\frac{3}{2}>1 \text{ (Conv.)}$$

$$\frac{u_n}{u_{n+1}}=\frac{2n+2}{2n+1}\times\frac{n+1}{n}$$

$$\lim\frac{u_n}{u_{n+1}}=\lim\frac{\cancel{n}\left(2+\frac{2}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)}\cdot\frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}}$$

$$=\frac{2\times 1}{2}$$

$$=1$$

Ratio test fails.

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \left(\frac{3+\frac{7}{n}}{3+\frac{3}{n}} \right) \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By Ratio test, $\sum u_n$ conv. for $\frac{1}{x} > 1$ i.e. $x < 1$

$\sum u_n$ div. for $\frac{1}{x} < 1$ i.e. $x > 1$

Test fails for $x = 1$

Prob: Test for convergence of the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} \cdot x^n, \quad x > 0$$

$$\rightarrow \quad u_n = \frac{3 \cdot 6 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} x^n$$

$$u_{n+1} = \frac{3 \cdot 6 \cdots 3n \cdot (3n+3)}{7 \cdot 10 \cdot 13 \cdots (3n+4)(3n+7)} \cdot x^{n+1}$$

$$= \lim \frac{4n}{3n+3}$$

$$= \lim \frac{4}{3+\frac{3}{n}} = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test

$\sum u_n$ converges (For $x=1$)

For $x=1$,

applying Raabe's test

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim n \cdot \left(\frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}\right)$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \times \frac{2n+3}{2n+1} \cdot \frac{1}{x^2}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{(2+\frac{2}{n})}{(2+\frac{1}{n})} \cdot \frac{(2+\frac{3}{n})}{(2+\frac{1}{n})} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

By Ratio test

$\sum u_n$ converges for $\frac{1}{x^2} > 1$

i.e. $x^2 < 1$

i.e. $0 < x < 1$

$\sum u_n$ div. for $\frac{1}{x^2} < 1$

i.e. $x > 1$

IAS 2023 ~~Prob~~: Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \cdot \frac{x^{2n+1}}{2n+1},$$

$x > 0$

$$\rightarrow u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \cdot \frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots 2n(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$= \lim \frac{n\cdot(6n+5)}{(2n+1)^2}$$

$$= \lim \frac{\cancel{n^2}\cdot\left(6+\frac{5}{n}\right)}{\cancel{n^2}\left(2+\frac{1}{n}\right)^2}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's test

Σu_n converges for $x=1$.

Test fails for $x=1$

Applying Raabe's test

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} - 1\right]$$

$$= \lim n.\left[\frac{\cancel{4n^2}+6n+\cancel{4n}+6-\cancel{4n^2}-1-\cancel{4n}}{(2n+1)2}\right]$$

$$\frac{u_n}{u_{n+1}} = \frac{n+\beta}{n+\alpha}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{1+\frac{\beta}{n}}{1+\frac{\alpha}{n}}$$

$$= 1$$

Ratio test fails.

Applying Raabe's test

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) =$$

<u>Prob</u>: Test for convergence of the series

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots$$

$$\rightarrow u_n = \frac{(1+\alpha)(2+\alpha)\cdots(n-1+\alpha)}{(1+\beta)(2+\beta)\cdots(n-1+\beta)}$$

$$u_{n+1} = \frac{(1+\alpha)\cdots(n+\alpha)}{(1+\beta)\cdots(n+\beta)}$$

By Raabe's test

$\sum u_n$ converge for

$\beta - \alpha > 1$

i.e. $\beta > 1 + \alpha$

$\sum u_n$ diverge for

$\beta - \alpha < 1$

i.e. $\beta < 1 + \alpha$

Test fails, when $\beta = 1 + \alpha$

In this case. Series is

$$\cancel{\frac{\alpha}{1+\alpha}} + \frac{1+\alpha}{2+\alpha} + \frac{1+\alpha}{3+\alpha} + \cdots$$

$$\lim n\left(\frac{n+\beta}{n+\alpha} - 1\right)$$

$$= \lim n\left[\frac{\cancel{n}+\beta-\cancel{n}-\alpha}{n+\alpha}\right]$$

$$= \lim \frac{n(\beta-\alpha)}{n+\alpha}$$

$$= \lim \frac{\beta-\alpha}{1+\frac{\alpha}{n}} = \beta - \alpha$$

By limit form of comparison ~~test~~, the given series diverges as $\sum \frac{1}{n}$ div.

$$= \sum \frac{1+\alpha}{(n+1)+\alpha} \sim \frac{1}{n}$$

$$\lim \frac{\frac{1+\alpha}{(n+1)+\alpha}}{\frac{1}{n}}$$

$$= \lim \frac{n(1+\alpha)}{n+1+\alpha}$$

$$= \lim \frac{1+\alpha}{1+\frac{1+\alpha}{n}} = 1+\alpha \neq 0$$

helpful when 'e' is present in $\frac{u_n}{u_{n+1}}$.

<u>prob</u>: Test for Convergence of the series

$1 + \frac{x}{\lfloor 1} + \frac{2^2 \cdot x^2}{\lfloor 2} + \frac{3^3 \cdot x^3}{\lfloor 3} + \cdots,$ $x > 0$

<u>IAS 2022</u>

Ignoring 1st term

$\rightarrow u_n = \frac{n^n \cdot x^n}{\lfloor n}$

<u>Logarithmic test</u>: If Σu_n is a +ve term series such that

$\lim n \cdot \log\left(\frac{u_n}{u_{n+1}}\right) = l$ then

(i) the series converges for $l > 1$

(ii) the series diverges for $l < 1$

<u>Note</u>: Logarithmic test is more

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{1}{x} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{x} \cdot \frac{1}{e}$$

$$= \frac{1}{ex}.$$

By Ratio test

$\sum u_n$ Conv. for $\frac{1}{ex} > 1$

i.e. $\frac{1}{e} > x$

i.e. $x < \frac{1}{e}$.

$\sum u_n$ div. $\frac{1}{ex} < 1$

i.e. $x > \frac{1}{e}$.

$$u_n = \frac{x^n \cdot n^n}{\lfloor n}$$

$$u_{n+1} = \frac{x^{n+1} (n+1)^{n+1}}{\lfloor n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x^n} \cdot n^n}{\cancel{\lfloor n}} \times \frac{\cancel{\lfloor n+1}}{\cancel{x^{n+1}}_{x} \, (n+1)^{n+\cancel{1}}}$$

$$= \frac{n^n}{x\,(n+1)^n}$$

$$= \lim n \cdot \left[1 + n \cdot \log\left(\frac{n+1}{n}\right)^{-1}\right]$$

$$= \lim n \cdot \left[1 - n \log\left(1 + \frac{1}{n}\right)\right]$$

$$= \lim n \cdot \left[\cancel{1} - n\left(\cancel{\frac{1}{n}} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots\right)\right]$$

$$= \lim n \cdot \left[\frac{1}{2n} - \frac{1}{3n^2} \cdots\right]$$

$$= \lim \left[\frac{1}{2} - \frac{1}{3n} \cdots\right]$$

$$= \frac{1}{2} < 1$$

<u>Div.</u>

Test fails.

when $x = \frac{1}{e}$.

Applying log. test.

$$\lim n \cdot \log\left(\frac{u_n}{u_{n+1}}\right)$$

$$= \lim n \cdot \log\left(e \cdot \left(\frac{n}{n+1}\right)^n\right)$$

$$= \lim n \cdot \left[\log e + n \cdot \log \frac{n}{n+1}\right]$$