KGS IAS

Maths Optional

By Dhrub Singh Sir

Linear D.E. of 2nd order with variable coeff.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

1st method: Changing of Dependent variable when one integral of C.F. is known.

Working rule: **Step I**: Change the diff. eqn. in standard form.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Step 2: Find one soln. $y = u(x)$ of the C.F. using the followings.

$\underline{m=1}$

$px + qx^2 = 0$ $\quad \underline{\underline{x}}$

$\Rightarrow \underline{p + qx = 0}$

$\underline{m=2}$

$2 + 2px + qx^2 = 0, \quad x^2$

Condition satisfied	one soln. of the C.F.
I: $a^2 + ap + q = 0$	e^{ax}
$\underline{a=1}$ $\quad 1 + p + q = 0$	e^{x}
$\underline{a=-1}$ $\quad 1 - p + q = 0$	e^{-x}
II: $m(m-1) + mpx + qx^2 = 0,$	x^m

Prob: Solve

$$x y'' - (2x-1)y' + (x-1)y = 0$$

Solⁿ:

$$\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0 \quad (1)$$

Comparing this with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q y = R$$

Step 3: General solⁿ. of the given diff. eqn. (1) is

$$\underline{y = u \cdot v}$$

where $v(x)$ is the solⁿ. of

$$\frac{d^2 v}{dx^2} + \left(\underbrace{P + \frac{2}{u}\frac{du}{dx}}\right)\frac{dv}{dx} = \underbrace{\frac{R}{u}}$$

General soln. of ① is given by

$y = e^x \cdot v$ where

v is the soln. of

$$\frac{d^2v}{dx^2} + \underbrace{\left(P + \frac{2}{u}\frac{du}{dx}\right)}\frac{dv}{dx} = 0 \quad \text{(ii)}$$

$$P + \frac{2}{u}\frac{du}{dx} = -\left(2 - \frac{1}{x}\right) + \frac{2}{\cancel{e^x}} \cdot \cancel{e^x}$$
$$= \frac{1}{x}$$

$P = -\left(2 - \frac{1}{x}\right)$, $Q = \left(1 - \frac{1}{x}\right)$

$R = 0$

$P + Q = -2 + \cancel{\frac{1}{x}} + 1 - \cancel{\frac{1}{x}}$

$P + Q = -1$

$\Rightarrow \underline{P + Q + 1 = 0}$

$y = e^x$ is an int of C.F.

$$z \cdot x = \int x \cdot 0 \, dx + c_1$$

$$\Rightarrow z \cdot x = c_1$$

$$\Rightarrow z = \frac{c_1}{x}$$

$$\Rightarrow \frac{du}{dx} = \frac{c_1}{x}$$

$$\Rightarrow du = \frac{c_1}{x} dx$$

int.

$$u = c_1 \log x + c_2$$

(ii) $\Rightarrow$

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0$$

$$\frac{dz}{dx} + \frac{1}{x} z = 0$$

$$z = \frac{du}{dx}$$

(say)

$$\text{If} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x} = x$$

Sol^n:

Comparing ① with

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

$P=-(1+x)$, $Q=x$, $R=x$

$$P+Q=-1-x+x$$
$$=-1$$
$$\Rightarrow P+Q+1=0$$

$\therefore$ $y=e^x$ is an integral of C.F.

$\therefore y=uv$

$=e^x\left[c_1\log x+c_2\right]$

is the general sol^n of ①

Prob: Solve

$$\frac{d^2y}{dx^2}-(1+x)\frac{dy}{dx}+xy=x \quad -①$$

(11) $\Rightarrow$

$$\frac{d^2v}{dx^2}+(1-x)\frac{dv}{dx}=\frac{x}{e^x}$$

i.e. $\frac{dz}{dx}+(1-x)z=xe^{-x}$ $\qquad z=\frac{dv}{dx}$

$$IF=e^{\int(1-x)\,dx}$$
$$=e^{x-\frac{x^2}{2}}$$

$y=e^{x}\cdot v$ is the general soln. of (1), where v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}\cdot \text{— (11)}$$

$$P+\frac{2}{u}\frac{du}{dx}=-(1+x)+\frac{2}{\cancel{e^x}}\cdot\cancel{e^x}$$
$$=-1-x+2$$
$$=1-x$$

$$\frac{dv}{dx} = -e^{-x} + c_1 e^{-x+\frac{x^2}{2}}$$

$$\Rightarrow dv = -e^{-x}dx + c_1 e^{-x+\frac{x^2}{2}} dx$$

Int.

$$v = e^{-x} + c_1 \int e^{-x+\frac{x^2}{2}} dx + c_2$$

$$\therefore y = uv$$

$$= e^{x}\left[e^{-x} + c_1 \int e^{-x+\frac{x^2}{2}} dx + c_2\right]$$

$$= \underline{1 + c_1 e^{x} \int e^{-x+\frac{x^2}{2}} dx + c_2 e^{x}}.$$

$$z \cdot e^{x-\frac{x^2}{2}} = \int e^{x-\frac{x^2}{2}} \cdot x \cdot e^{-x} dx + c_1$$

$$\Rightarrow z \cdot e^{x-\frac{x^2}{2}} = \int e^{-\frac{x^2}{2}} \cdot x\, dx + c_1$$

$$= -\int e^{-\frac{x^2}{2}} \cdot (-x) dx + c_1$$

$$= -e^{-\frac{x^2}{2}} + c_1$$

$$\Rightarrow z = -e^{-x} + c_1 e^{-x+\frac{x^2}{2}}$$

$y = \cos x \cdot v$ is a gen. solⁿ of ①

where v is given by

$$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2 v}{dx^2} + \left(\frac{2 \times (-\sin x)}{\cos x}\right)\frac{dv}{dx} = \sec^2 x \quad (11)$$

$$\Rightarrow \frac{d^2 v}{dx^2} - 2\tan x \frac{dv}{dx} = \sec^2 x$$

Prob: Solve

$$\frac{d^2 y}{dx^2} + y = \sec x$$ given that

$\cos x$ is a part of C.F. (an integral)

Solⁿ:

$$\frac{d^2 y}{dx^2} + y = \sec x \quad — (1)$$

$P = 0, \quad Q = 1, \quad R = \sec x$

$\therefore$

$$z\cdot\cos^2 x = \int \cos^2 x \cdot \sec^2 x\,dx + c_1$$

$$z\cdot\cos^2 x = x + c_1$$

$$z = x\sec^2 x + c_1\sec^2 x$$

$$\Rightarrow \frac{du}{dx} = x\sec^2 x + c_1\sec^2 x$$

Int

$$u = \int x\sec^2 x\,dx + c_1\int \sec^2 x\,dx + c_2$$

$$= x\tan x - \int \tan x\,dx + c_1\tan x + c_2$$

$$= (x+c_1)\tan x + \log\cos x + c_2$$

$$\frac{dz}{dx} - 2\tan x\, z = \sec^2 x$$

$$z = \frac{du}{dx}$$

$$IF = e^{-2\int \tan x\,dx}$$

$$= e^{2\int \frac{-\sin x}{\cos x}dx}$$

$$= e^{2\log\cos x}$$

$$= e^{\log(\cos x)^2}$$

$$= \cos^2 x$$

Solⁿ. $\dfrac{d^2y}{dx^2} - 2\dfrac{(1+x)}{x}\dfrac{dy}{dx} + 2\dfrac{(1+x)}{x^2}y = x$

Comparing,

$P = -2\left(1+\frac{1}{x}\right)$, $Q = 2\left(\frac{1}{x^2}+\frac{1}{x}\right)$,

$R = x$

$P + Qx$

$= -2\cancel{\left(1+\frac{1}{x}\right)} + \cancel{2\left(1+\frac{1}{x}\right)}$

$= 0$

$\therefore y = x$ is an integral of C.F of (1)

$\therefore y = \cos x\left[(x+c_1)\tan x + \log\cos x + c_2\right]$

gen. solⁿ. of (1)

Prob: Solve

$x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3.$

$\frac{dz}{dx} - 2z = 1$, $\quad z = \frac{du}{dx}$ (say)

$IF = e^{\int -2\,dx}$

$= e^{-2x}$

$\therefore z \cdot e^{-2x} = \int e^{-2x} \cdot 1\,dx + c_1$

$\Rightarrow z \cdot e^{-2x} = -\frac{1}{2} e^{-2x} + c_1$

$\Rightarrow z = -\frac{1}{2} + c_1 e^{2x}$

$\Rightarrow \frac{du}{dx} = -\frac{1}{2} + c_1 e^{2x}$

$y = x \cdot v$ is the gen. sol^n of (1), where v is given by

$$\frac{d^2u}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{du}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2u}{dx^2} + \left[-2\left(1+\frac{1}{x}\right) + \frac{2}{x}\right]\frac{du}{dx} = 1$$

$$\Rightarrow \frac{d^2u}{dx^2} - 2\frac{du}{dx} = 1$$

changing the dependent variable and removal of the 1st derivative.

Working rule:

Step 1: write the given eqn. in standard form.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Step 2: To remove the 1st derivative, choose $u = e^{-\frac{1}{2}\int P\,dx}$

$$\Rightarrow dv = -\frac{1}{2}dx + c_1 e^{2x}dx$$

int.

$$v = -\frac{1}{2}x + c_1\frac{1}{2}e^{2x} + c_2$$

$$\therefore y = x\left[-\frac{1}{2}x + c_1\frac{1}{2}e^{2x} + c_2\right]$$

is the gen. soln. of (1)

Prob: Solve

$$y'' - 4x y' + (4x^2-1) y = -3e^{x^2} \sin 2x \quad \text{--- (1)}$$

Solⁿ: Comparing,

$$P = -4x, \quad Q = 4x^2 - 1$$

$$R = -3e^{x^2} \cdot \sin 2x$$

let $u = e^{-\frac{1}{2}\int -4x\,dx}$

$$= e^{2 \cdot \frac{x^2}{2}} = e^{x^2}$$

Step 3: $y = uv$ is the gen. solⁿ of (1), v is given by

$$\underbrace{\frac{d^2 v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right) \cdot v}_{\text{Normal form}} = \frac{R}{u}.$$

$= 4\cancel{x^2} - 1 - 4\cancel{x^2} + 2$

$= 1$

(11) $\Rightarrow$

$$\frac{d^2 v}{dx^2} + v = -3\sin 2x$$

i.e. $(D^2 + 1)v = -3\sin 2x$

A-E. $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$y = e^{x^2} \cdot v$ is the general soln. of (I), v is given by

$$\frac{d^2 v}{dx^2} + \underbrace{\left(\varphi - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)}v = \frac{R}{u} \quad \text{—(11)}$$

$$\varphi - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 4x^2 - 1 - \frac{1}{4}(-4x)^2 - \frac{1}{2}(-4)$$

$v = c_1\cos x + c_2\sin x + \sin 2x$

$\therefore y = e^{x^2}\left[c_1\cos x + c_2\sin x + \sin 2x\right]$

is the gen. sol^n. of (1).

$\therefore CF = c_1\cos x + c_2\sin x$

$$PI = \frac{1}{D^2+1}(-3\sin 2x)$$

$$= -3\frac{1}{D^2+1}\sin 2x$$

$$= \cancel{-3}\frac{1}{\cancel{-4+1}}\sin 2x$$

$$= \sin 2x$$

$$\frac{d^2y}{dx^2} + y + 2\tan x \frac{dy}{dx} + 2y\tan^2 x = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2\tan x \frac{dy}{dx} + (1 + 2\tan^2 x)y = 0 \quad \text{(1)}$$

Comparing,

$P = 2\tan x$, $Q = 1 + 2\tan^2 x$

$R = 0$

Prob: Solve

$$\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = 0$$

Solⁿ.

$$\cot x \cdot \frac{d^2y}{dx^2} + y\cot x + 2\frac{dy}{dx} + 2y\tan x = 0$$

v is given by

$$\frac{d^2 v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)v = 0$$

$$\Rightarrow \frac{d^2 v}{dx^2} + \left(1+\cancel{2}\tan^2 x - \cancel{\tan^2} x - \frac{1}{\cancel{2}} \times \cancel{2}\sec^2 x\right)v = 0$$

$$\Rightarrow \frac{d^2 v}{dx^2} + \left(\sec^2 x - \sec^2 x\right)v = 0$$

$$\Rightarrow \frac{d^2 v}{dx^2} = 0$$

Let $u = e^{-\frac{1}{\cancel{2}}\int \cancel{2}\tan x\, dx}$

$= e^{-\int \tan x\, dx}$

$= e^{\int \frac{-\sin x}{\cos x}\, dx}$

$= e^{\log \cos x}$

$= \cos x$

$y = u \cdot v$ is the gen. solⁿ of (1),

$\therefore y = \cos x (c_1 x + c_2)$

is the gen. solⁿ. of ①.

Prob: Solve

HW

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0$$

Ans: $y = (c_1 \sin\sqrt{6}\,x + c_2 \cos\sqrt{6}\,x).$

$\sec x.$

$$\frac{d}{dx}\left(\frac{dv}{dx}\right) = 0$$

int w.r.t. x

$$\frac{dv}{dx} = c_1$$

$\Rightarrow dv = c_1 dx$

Int

$v = c_1 x + c_2$