Chapter
02

Sequence And Series

CONTENT

1. INTRODUCTION:

A succession of numbers t_1, t_2,.....t_n formed according to some definite rule is called sequence. "A sequence is a function of natural numbers with codomain as the set of Real numbers or complex numbers"

Domain of sequence = N

If Range of sequence $\subseteq$ R $\Rightarrow$ Real sequence

If Range of sequence $\subseteq$ C $\Rightarrow$ Complex sequence

Sequence is called finite or infinite depending upon its having number of terms as finite or infinite respectively.

Ex. : 2, 3, 5, 7, 11, ... is a sequence of prime numbers. It is an infinite sequence.

A progression is a sequence having its terms in a definite pattern.

Ex.: 1, 4, 9, 16, ... is a progression as each successive term is obtained by squaring the next natural number.

However, a sequence may not always have an explicit formula of n^{th} term.

Series is constructed by adding or subtracting the terms of a sequence

Ex. 2 + 4 + 6 + 8 + ...

The term at n^{th} place is denoted by T_n and is called general term of a sequence or progression or series.

Progression can be classified into 5 parts as-

(i) Arithmetic Progression (A.P.)
(ii) Geometric Progression (G.P.)
(iii) Harmonic Progression (H.P.)
(iv) Arithmetic Geometric Progression (A.G.P.)
(v) Miscellaneous Progression

2. ARITHMETIC PROGRESSION (A.P.)

Arithmetic Progression is defined as a series in which difference between any two consecutive terms is constant throughout the series. This constant difference is called common difference. If 'a' is the first term and 'd' is the common difference, then an A.P. can be written as a + (a+ d) + (a + 2d) + (a + 3d) + ...

2.1 General term/n^{th} term/Last term of A.P. :

It is given by $\mathbf{T_n = a + (n - 1)d}$

Where a = first term, d = common difference and n = position of the term which we require.

Note : (i) If d > 0 $\Rightarrow$ Increasing A.P.

If d < 0 $\Rightarrow$ Decreasing A.P.

If d = 0 $\Rightarrow$ All the term remains same

(ii) nth term from end is given by

$= T_m - (n-1)\,d$

or $= (m - n + 1)^{th}$ term from beginning, where m is total no. of terms.

2.2 Sum of n terms of an A.P.

The sum of first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}[a + T_n]$$

Note: (i) If sum of first n terms S_n of a sequence is given then general term $T_n = S_n - S_{n-1}$, where S_{n-1} is sum of first (n–1) terms of A.P.

(ii) Common difference of A.P. is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term or first term of an A.P.

(iii) The sum of infinite terms of an A.P. is ∞ if $d > 0$ and $-\infty$ if $d < 0$.

DETECTIVE MIND

- Sum of n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A
- n^{th} term of an A.P. is of the form An + B i.e. a linear expression in n, in such a case the coefficient of n is the common difference of the A.P. i.e. A

2.3 Highlights of an A.P. :

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(iii) Three numbers in A.P. can be taken as $a - d, a, a + d$.

Four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$.

Five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$;

six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it.

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}),\ k < n. \text{ For } k = 1,\ a_n = \left(\frac{1}{2}\right)(a_{n-1} + a_{n+1});$$

$$\text{For } k = 2,\ a_n = \left(\frac{1}{2}\right)(a_{n-2} + a_{n+2}) \text{ and so on.}$$

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an A.P.

(vii) The sum and difference of corresponding terms of two A.P.'s is an A.P.

(3) Arithmetic Mean (A.M.)

If three or more than three terms are in A.P., then the numbers lying between first and last term are known as Arithmetic Means between them. i.e. The A.M. between the two given quantities a and b is A, so that a, A, b are in A.P.

i.e. $A - a = b - A \Rightarrow$ **A =** $\frac{a+b}{2}$

Note : A.M. of any n positive numbers $a_1, a_2 ......a_n$ is

$$A = \frac{a_1 + a_2 + a_3 + ...a_n}{n}$$

n AM's between two given numbers

If in between two numbers 'a' and 'b' we have to insert n AM, $A_1, A_2, ...A_n$ then a, $A_1, A_2, A_3....A_n$, b will be in A.P. The series consist of (n+2) terms and the last term is b and first term is a.

$\Rightarrow$ $a + (n + 2 - 1)\, d = b$

$\Rightarrow$ $d = \frac{b-a}{n+1}$

$A_1 = a + d$, $A_2 = a + 2d, ...\ A_n = a + nd$ or $A_n = b - d$

DETECTIVE MIND

- Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$, where $A = \frac{a+b}{2}$

SOLVED EXAMPLES

Example: 1 In an A.P. if $a_2 + a_5 - a_3 = 10$ and $a_2 + a_9 = 17$ then find the 1st term and the common difference.

Solution: In an A.P.

Let a_1 = first term & d = common difference

$a_2 + a_5 - a_3 = 10$

$a_1 + d + a_1 + 4d - (a_1 + 2d) = 10$

$a_1 + 3d = 10$ ...(i)

$a_2 + a_9 = 17$

$a_1 + d + a_1 + 8d = 17$

$2a_1 + 9d = 17$ ...(ii)

On solving (i) & (ii) we get

$a_1 = 13$ $\quad d = -1$

Example: 2 The first term of an A.P. is 5, the last is 45 and the sum is 400. Find the number of terms and the common difference.

Solution: $a = 5, \quad \ell = 45$

Let common difference = d

$a + (n - 1)d = 45$

$S_n = 400$

$\frac{n}{2}[a+\ell] = 400$

$\frac{n}{2}[5+45] = 400$

$n = 16$

$\therefore \quad a + (n-1)d = 45$

$5 + 15d = 45$

$d = \frac{8}{3}$

Example: 3 Solve the equation (x + 1) + (x + 4) + (x + 7) + … … + (x + 28) = 155.

Solution: We have, (x + 1) + (x + 4) + … … + (x + 28) = 155

Let n be the number of terms in the A.P. on L.H.S. Then, $\begin{cases} x+28=(x+1)+(n-1)(3) \\ \Rightarrow n=10 \end{cases}$

$\therefore \quad (x + 1) + (x + 4) + \dots \dots + (x + 28) = 155$

$\Rightarrow \quad \frac{10}{2}\left[(x+1)+(x+28)\right]=155$

$\Rightarrow \quad x = 1$

Example: 4 How many terms of the sequence, $20 + 19\frac{1}{3} + 18\frac{2}{3} + \dots\dots\dots$ must be taken so that there sum is 300. Explain the reason of double answer.

Solution: $20 + 19\frac{1}{3} + 18\frac{2}{3} + \dots\dots\dots$

$a = 20, \qquad d = \frac{-2}{3}$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$300 = \frac{n}{2}\left[2\times 20+(n-1)\left(\frac{-2}{3}\right)\right]$

$300 = 20n - \frac{n(n-1)}{3}$

$n^2 - 61n + 900 = 0$

$n = 25, 36$

We got two values of n because when n = 25, all terms will be positive but when n = 36, terms will become negative.

Example: 5 The sum of n terms of two A.P.'s are in the ratio of 7n + 1 : 4n + 27, find the ratio of their 11th terms

Solution: Let the two series be $a_1, a_1 + d_1, a_1 + 2d_1$ ……. & $a_2, a_2 + d_2, a_2 + 2d_2$, … …

$S_1 = \frac{n}{2}\left[2a_1 + (n-1)d_1\right]$

$S_2 = \frac{n}{2}\left[2a_2 + (n-1)d_2\right]$,

$\therefore \frac{S_1}{S_2} = \frac{7n+1}{4n+27}$

$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$

$$\frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$,

$\therefore T_{11} = a + 10d$

So, to find ratio of 11th term we should Put $\frac{(n-1)}{2} = 10$

$\Rightarrow n = 21$

$\therefore \frac{S_1}{S_2} = \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{7(21)+1}{4(21)+27} = \frac{4}{3}$

Example: 6 The sum of first 3 terms of an A.P. is 27 and the sum of their squares is 293. Find absolute value fo S_{15}.

Solution: Let the terms be

$a - d, a, a + d$

$3a = 27$

$a = 9$

$(a - d)^2 + a^2 + (a + d)^2 = 293$

$3a^2 + 2d^2 = 293$

$2d^2 = 50$

$d = \pm 5$

When d = 5 then A.P. is 4, 9, 14, ……….

$\therefore \quad S_{15} = \frac{15}{2}[2 \times 4 + 14 \times 5] = 585$

When d = –5 then A.P. is 14, 9, 4,……….

$\therefore \quad S_{15} = \frac{15}{2}[2 \times 14 + 14 \times (-5)] = -315$

$\therefore \quad |S_{15}| = 315.$

Example: 7 Find the nature and 30th term of the sequence whose sum to n terms is $5n^2 + 2n$.

Solution: $S_n = 5n^2 + 2n$

$S_{n-1} = 5(n-1)^2 + 2(n-1)$

$T_n = S_n - S_{n-1} = 5(2n-1) + 2 = 10n - 3$

Series = 7, 17, 27, 37 ……

Series is A.P.

$T_{30} = 10 \times 30 - 3 = 297.$

Example: 8 If a^2, b^2, c^2 are in A.P. then prove that

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Solution: (i) Let $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\frac{2}{c+a} = \frac{1}{a+b} + \frac{1}{b+c}$

$2(a + b)(b + c) = (c + a)(2b + a + c)$

$2b^2 = a^2 + c^2$

Hence, a^2, b^2, c^2 are in A.P.

So if a^2, b^2, c^2 are in A.P., then

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(ii) If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Multiply each term with a + b + c

$\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$

Subtract 1 from each term

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Example: 9 Insert 20 AM's between 4 and 67.

Solution: a = 4, b = 67, n = 20, $d = \frac{b-a}{n+1} = \frac{67-4}{21} = 3$

$A_1 = a + d = 4 + 3 = 7$

$A_2 = a + 2d = 4 + 6 = 10$

$A_3 = 13$, $A_4 = 16$

$A_{20} = a + 20d = 4 + 20\times3 = 64$.

So, required solution 4, 7, 10, 13, 16, ……… 67.

Example: 10 If p arithmetic means are inserted between 5 and 41 so that the ratio $\frac{A_3}{A_{p-1}} = \frac{2}{5}$ then find the value of p.

Solution: a = 5, b = 41, number of arithmetic means = p

$d = \frac{b-a}{n+1} = \frac{36}{p+1}$

$A_3 = a + 3d = 5 + 3\left(\frac{36}{p+1}\right) = \frac{5p+5+108}{p+1}$

$A_{p-1} = 41 - 2d = 41 - 2\left(\frac{36}{p+1}\right) = \frac{41p-31}{p+1}$

Now given $\frac{A_3}{A_{p-1}} = \frac{2}{5} \Rightarrow \frac{5p+113}{41p-31} = \frac{2}{5}$

25p + 565 = 82p – 62 $\Rightarrow$ p = 11

Example: 11 If eleven A.M.'s are inserted between 28 and 10, then find the number of integral A.M.'s.

Solution: Assume A_1, A_2, A_3 ……,A_{11} be the eleven A.M.'s between 28 and 10, so 28, A_1, A_2, ……A_{11}, 10 are in A.P. Let d be the common difference of the A.P. The number of terms in 13. Now,

$d = \frac{b-a}{n+1} = \frac{10-28}{11+1} = -\frac{3}{2}$, Hence, the number of integral A.M.'s is 5.

Example: 12 If sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3, respectively, then find the value of $\frac{S_3}{S_2 - S_1}$.

Solution: Let a be the first term and d be the common difference of the given A.P. Then, sum of n terms is

$S_1 = \frac{n}{2}\left[2a+(n-1)d\right]$

Sum of 2n terms,

S_2 = Sum of 2n terms = $\frac{2n}{2}\left[2a+(2n-1)d\right]$

Sum of 3n terms,

$S_3 = \frac{3n}{2}\left[2a+(3n-1)d\right]$

Now, $S_2 - S_1 = \frac{2n}{2}\left[2a+(2n-1)d\right] - \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{n}{2}\left[2\{2a+(2n-1)d\}-\{2a+(n-1)\}d\right] \quad = \frac{n}{2}\left[2a+(3n-1)d\right]$

$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}[2a + (3n - 1)d] = S_3$

$\Rightarrow \quad \frac{S_3}{S_2 - S_1} = 3$

Example: 13 The numbers $t(t^2 + 1)$, $-\frac{t^2}{2}$ and 6 are three consecutive terms of an A.P. If t be real, then find the the next two term of A.P.

Solution: By using $2b = a + c$

$-t^2 = t^3 + t + 6$

or $t^3 + t^2 + t + 6 = 0$

or $(t + 2)(t^2 - t + 3) = 0$

$\therefore \quad t^2 - t + 3 \neq 0 \Rightarrow \qquad t = -2$

the given numbers are – 10, – 2, 6

which are in an A.P. with d = 8. The next two numbers are 14, 22

Example: 14 If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution: As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 2a_3 + 2a_3 + a_3 = 5a_3 = 10$

Example: 15 If $a(b + c), b(c + a), c(a + b)$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Solution: $\because \quad a(b + c), b(c + a), c(a + b)$ are in A.P.,

subtract $ab + bc + ca$ from each

$\Rightarrow \quad -bc, -ca, -ab$ are in A.P.

divide by $-abc$,

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Example: 16 If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P. then prove that $\frac{1}{a}$, b $\frac{1}{c}$ are in A.P.

Solution: $\therefore$ $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P.

$b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b \quad \Rightarrow \quad \frac{-a\left(b^2+1\right)}{1-ab} = \frac{c\left(1+b^2\right)}{1-bc}$

$\Rightarrow$ $-a + abc = c - abc$

$\Rightarrow$ $a + c = 2abc$

divide by ac,

$\frac{1}{c}+\frac{1}{a} = 2b \quad \Rightarrow \quad \frac{1}{a}$, b, $\frac{1}{c}$ are in A.P.

Example: 17 If a, b, c, d, e, f are A. M's between 2 and 12, then find a + b + c + d + e + f.

Solution: Sum of A.M.^{s} = 6 single A.M. = $\frac{6(2+12)}{2}$ = 42

---

4. GEOMETRIC PROGRESSION (G.P.) :

In a sequence, if each term (except the first non zero term) bears the same constant ratio with its immediately preceding term then the series is called a G.P. and the constant ratio is called the common ratio.

Standard appearance of a G.P. is

$a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots, ar^{n-1}$.

4.1 General term/nth term/Last term of G.P. :

It is given by $T_n = a.r^{n-1}$

Where a = first term, r = common ratio and n = position of the term which we required.

4.2 Sum of n terms of a G.P.

The sum of first n terms of an A.P. is given by

$S_n = \frac{a(1-r^n)}{1-r}$, when r < 1

or $S_n = \frac{a(r^n-1)}{r-1}$, when r >1

and $S_n = an$ when r = 1

4.3 Sum of an infinite G.P.

The sum of an infinite G.P. with first term 'a' and common ratio 'r' ($-1 < r < 1$ i.e. $|r| < 1$) is $S_\infty = \dfrac{a}{1-r}$

Note : If $r \geq 1$ then $S_\infty \to \infty$

4.4 Highlights of G.P. :

(i) Any 3 consecutive terms of a GP can be taken as $\frac{a}{r}$, a, ar.

any 4 consecutive terms of a GP can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 & so on.

(ii) In a finite G.P. the product of the terms equidistant from the beginning and the end are equal.

$a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = ...$

(iii) If a, b, c are in GP $\Rightarrow b^2 = ac$.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non–zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3,........$ and $b_1, b_2, b_3,.........$ are two G.P's with common ratio r_1 and r_2 respectively, then the sequence $a_1b_1, a_2b_2, a_3b_3, ...$ is also a G.P. with common ratio $r_1 r_2$.

(vi) If $a_1, a_2, a_3,..........$are in G.P. where each $a_i > 0$, then $\log a_1, \log a_2, \log a_3,..........$are in A.P. and its converse is also true.

5. GEOMETRIC MEAN (G.M.) :

If three or more than three terms are in G.P. then all the numbers lying between first and last term are called Geometrical Means between them. i.e.

If the G.M. between two given quantities a and b is G, then a, G, b, are in G.P.

i.e. $\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$

Note : (i) G.M. of any n positive numbers $a_1, a_2, a_3, ... a_n$ is $(a_1.a_2.a_3.....a_n)^{1/n}$.

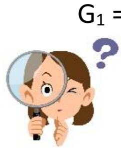
(ii) If a and b are two numbers of opposite signs, then GM between them does not exist.

n GM's between two given numbers

If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2....G_n$ then a, $G_1, G_2, ... G_n$, b will be in GP. The series consist of (n+2) terms and the last term is b and first term is a.

$\Rightarrow ar^{n+2-1} = b \qquad \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$G_1 = ar$, $G_2 = ar^2$ ......$G_n = ar^n$ or $G_n = b/r$



DETECTIVE MIND

Product of n GM's inserted between 'a' and 'b' is equal to n^{th} power of the single GM between 'a' and 'b'

i.e. $\prod_{r=1}^{n} G_r = (G)^n$, where $G = \sqrt{ab}$

SOLVED EXAMPLES

Example: 18 In a G.P. if t_3 = 2 and t_6 = $-\frac{1}{4}$ find t_{10}.

Solution: $t_3 = 2, \quad t_6 = -\frac{1}{4} \quad \Rightarrow \quad ar^2 = 2, \ ar^5 = -\frac{1}{4}$

$\frac{1}{r^3} = -8 \quad \Rightarrow \quad \frac{1}{r} = -2$

$r = \frac{-1}{2} \quad \Rightarrow \quad a = 8$

$t_{10} = ar^9 = 8\left(\frac{-1}{2}\right)^9 = \frac{-1}{64}$

Example: 19 Find the four successive terms of a G.P. of which the 2nd term is smaller than the first by 35 and the 3rd term is larger than the 4th by 560.

Solution: Let the four terms be

a, ar, ar^2, ar^3

$a - ar = 35$

$ar^2 - ar^3 = 560$

$\frac{a(1-r)}{ar^2(1-r)} = \frac{35}{560}$

$r^2 = 16 \quad \Rightarrow \quad r = \pm 4$

Since 2nd term is less than 1st so $r = -4 \Rightarrow a = \frac{35}{1-r} = \frac{35}{1+4} = 7$

terms are 7, –28, 112, –448

Example: 20 If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y and z respectively then find the value of $x^{q-r} . y^{r-p} . z^{p-q}$.

Solution: Let a and k be the first term and common ratio of the G.P. respectively.

$x = a k^{p-1}$

$y = a k^{q-1}$

$z = a k^{r-1}$

$x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$

$= \left(\frac{x}{z}\right)^q \cdot \left(\frac{y}{x}\right)^r \cdot \left(\frac{z}{y}\right)^p = \left(k^{p-r}\right)^q \left(k^{q-p}\right)^r \left(k^{r-q}\right)^p = k^{pq-qr+qr-rp+rp-pq} = k^0 = 1$

Example: 21 Determine the number of terms in a G.P. if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

Solution: $a_1 = 3$

$a_1 r^{n-1} = 96$

$r^{n-1} = 32$

$\frac{a(1-r^n)}{1-r} = 189 \quad \Rightarrow \quad \frac{3(1-32r)}{1-r} = 189$

$93r = 186$

$r = 2$

$2^{n-1} = 32$

$n = 6$

Example: 22 Find the least value of $n(n \in N)$ for which sum of the series $1 + 3 + 3^2 + 3^3 + \ldots\ldots\ldots$ upto n terms exceeds 9000.

Solution: $S_n = 1 + 3 + 3^2 + \ldots\ldots + n$ term

$S_n = \frac{1\cdot(3^n - 1)}{3-1} > 9000$

$3^n - 1 > 18000,\ \ 3^n > 18001$

$n \geq 9$

Least value of $n = 9$.

Example: 23 The sum of an infinite number of terms of a G.P. is 15 and the sum of their squares is 45. Find the series.

Solution: $a + ar + ar^2 + ar^3 + \ldots\ldots = 15$

$\frac{a}{1-r} = 15$ ...(i)

$a^2 + (ar)^2 + (ar^2)^2 + \ldots\ldots = 45$

$\frac{a^2}{(1-r^2)} = 45$ ...(ii)

$\frac{(1-r)^2}{1-r^2} = \frac{45}{225} \Rightarrow \frac{1-r}{1+r} = \frac{1}{5}$

$5 - 5r = 1 + r$

$r = 2/3$

$a = 15\left(1 - \frac{2}{3}\right) = 5$

$5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27} \ldots\ldots$

Example: 24 If $x = a + \frac{a}{r} + \frac{a}{r^2} + \ldots\ldots\infty$, $y = b - b/r + b/r^2 - \ldots\ldots \infty$ and $z = c + c/r^2 + c/r^4 + \ldots\ldots \infty$, then prove that $\frac{xy}{z} = \frac{ab}{c}$.

Solution: We have,

$x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$

$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$

$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{(r^2-1)}$

$xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$

$$\therefore \quad \frac{xy}{z}=\left[\frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}\right]=\frac{ab}{c}$$

Example: 25 The sum of first 3 consecutive terms of a G.P. is 19 and their product is 216. Find S_{20}, also compute S_∞ if it exist.

Solution: Let the 3 consecutive terms be

$\frac{a}{r}$, a, ar

$\therefore \frac{a}{r} + a + ar = 19$

and $\frac{a}{r}\cdot a\cdot ar = 216$

$\Rightarrow a^3 = 216$

$\Rightarrow a = 6$

$\therefore \frac{6}{r} + 6 + 6r = 19$

$6(1 + r + r^2) = 19r$

$6r^2 - 13r + 6 = 0$

$(3r - 2)(2r - 3) = 0$

$r = \frac{2}{3}, \frac{3}{2}$

When $r=\frac{3}{2}, S_{20}=8\left[\left(\frac{3}{2}\right)^{20}-1\right]$ and S_∞ does not exist.

When $r = \frac{2}{3}$, $S_{20}=27\left(1-\left(\frac{2}{3}\right)^{20}\right)$ and $S_\infty=\frac{9}{1-\frac{2}{3}}=27$.

Example: 26 If a_n = n^{th} term of G.P. and

$a_1 + a_2 + a_3 = 13$ and $a_1^2+a_2^2+a_3^2=91$ then find a_{50}.

Solution: Given,

$a_1 + a_2 + a_3 = 13$, Let $a_1 = \frac{a}{r}$, $a_2 = a$, $a_3 = ar$

$\frac{a}{r} + a + ar = 13 \quad \Rightarrow \quad a\left(\frac{1}{r}+1+r\right)=13$

$r+\frac{1}{r}=\frac{13}{a}-1$

$r+\frac{1}{r}=\frac{13-a}{a}$ ______(1)

$$r^2+\frac{1}{r^2}+2=\left(\frac{13-a}{a}\right)^2$$

Now given

$$a_1^2+a_2^2+a_3^2=91$$

$$a^2\left(\frac{1}{r^2}+1+r^2\right)=91$$

$$a^2\left(\frac{(13-a)^2}{a^2}-1\right)=91$$

$(13-a)^2-a^2=91 \quad \Rightarrow \quad 13^2-26a=91$

$26a = 169 - 91 \quad \Rightarrow \quad 26a = 78$

$a = 3$

$\Rightarrow 3r^2 + 3 = 10r$ (Using (1))

$3r^2 - 9r - r + 3 = 0$

$r = 3, \frac{1}{3}$

When a = 3 and r = 3 then $a_{50} = 3\cdot 3^{49} = 3^{50}$.

And when a = 3 and r = $\frac{1}{3}$ then $a_{50}=3\cdot\left(\frac{1}{3}\right)^{49}=\left(\frac{1}{3}\right)^{48}$.

Example: 27 Use infinite series to compute the rational number corresponding to $0.4\overline{23}$

Solution: $x = 0.4\overline{23}$

$= 0.4 + 0.023 + 0.00023 + \ldots\ldots$

$= 0.4 + \frac{23}{10^3}+\frac{23}{10^5}+\frac{23}{10^7}+\ldots\ldots$

$= \frac{4}{10}+\frac{23}{10^3}\left(1+\frac{1}{10^2}+\frac{1}{10^4}+\ldots\ldots\right)$

$= \frac{4}{10}+\frac{23}{10^3}\left(\frac{1}{1-1/100}\right)$

$x = \frac{4}{10}+\frac{23}{990}=\frac{419}{990}$

Example: 28 Insert 4 GM's between 5 and 160.

Solution: $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}=\left(\frac{160}{5}\right)^{\frac{1}{5}} = (32)^{1/5}=2$

$G_1 = ar = 10$

$G_2 = 20$

$G_3 = 40$

$G_4 = 80$

Example: 29 If AM between two positive numbers a and b is 15 and GM between a and b is 9. Find the numbers.

Solution: $\therefore \frac{a+b}{2} = 15, \quad a + b = 30$

and $\sqrt{ab}=9$

$\Rightarrow \sqrt{a(30-a)} = 9$

$\Rightarrow$ a(30 – a) = 81

$\Rightarrow$ $a^2 - 30a + 81 = 0$

$\Rightarrow$ a = 3, 27,   Hence two nos. are 3, 27.

Example: 30 If the 10^{th}, 15^{th}, 25^{th} terms of an A.P. are in G.P. then find the common ratio of the G.P.

Solution: Let the first term and common ratio of the AP be a and d respectively.

$T_{10} = a + 9d$, $T_{15} = a + 14d$, $T_{25} = a + 24d$.

$R = \frac{T_{15}}{T_{10}} = \frac{T_{25}}{T_{15}}$

$R = \frac{a+14d}{a+9d} = \frac{a+24d}{a+14d} = \frac{a+14d-a-24d}{a+9d-a-14d} = 2$

Example: 31 The n^{th} term of the series 3, $\sqrt{3}$, 1 , ------ is $\frac{1}{243}$, then find n.

Solution: $3.\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243} \Rightarrow n = 13$

Example: 32 The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution: Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow$ $r = \frac{r^2}{1-r}$ $\Rightarrow$ $r = \frac{1}{2}$,

Hence series is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\infty$

Example: 33 Let three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is :

Solution: Three number in G.P. are $\frac{a}{r}$, a, ar

then $\frac{a}{r}$, 2a, ar are in A.P. as given.

$\therefore$ $2(2a) = a\left(r + \frac{1}{r}\right)$

$\Rightarrow$ $r^2 - 4r + 1 = 0$ or $r = 2 \pm \sqrt{3}$

or $r = 2 + \sqrt{3}$ as r > 1 for an increasing G.P.

6. HARMONIC PROGRESSION (H.P.)

A sequence is said to be in H.P if the reciprocals of its terms are in A.P.. If the sequence $a_1, a_2, a_3, ..., a_n$ is in H.P. then $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ is in A.P.

Note : (i) Here we do not have the formula for the sum of the n terms of an H.P For H.P. whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

(ii) If a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

(iii) If a, b, c are in A.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{a}$

(iv) If a, b, c are in G.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{b}$

7. HARMONIC MEAN (H.M.):

If a, b, c are in H.P., b is called as the H.M. between a & c, then $b = \frac{2ac}{a+c}$

If a_1, a_2 , ... a_n are 'n' non-zero numbers then H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

8 RELATION BETWEEN A.M., G.M. & H.M.

A, G, H are AM, GM and HM respectively between two numbers 'a' and 'b' then

$A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$

(i) Consider $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0$

So $A \geq G$

In the same way $G \geq H \Rightarrow A \geq G \geq H$

(ii) Consider $A.H = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2 \Rightarrow G^2 = A.H.$

SOLVED EXAMPLES

Example: 34 If x > 0, y > 0, z > 0 then prove that $(x + y)(y + z)(z + x) \geq 8xyz$

Solution: $(x + y)(y + z)(z + x)$

$\frac{x+y}{2} \geq \sqrt{xy}$ (A.M. ≥ G.M.)

$\frac{y+z}{2} \geq \sqrt{yz}$

$\frac{z+x}{2} \geq \sqrt{zx}$

$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz$

$(x + y)(y + z)(z + x) \geq 8xy$

Example: 35 If a + b + c = 3 and a, b, c are positive then prove that $a^2b^3c^2 \le \frac{3^{10}\cdot 2^4}{7^7}$

Solution: a + b + c = 3

We can write it as

$$\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}=3$$

Now A.M. $\ge$ G.M.

$$\frac{\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}}{7}\ge\left(\frac{a^2}{4}\frac{b^3}{27}\frac{c^2}{4}\right)^{1/7} \Rightarrow \frac{3}{7}\ge\left(\frac{a^2b^3c^2}{2^4\times 3^3}\right)^{1/7}$$

$$a^2b^3c^2\le\frac{3^{10}\cdot 2^4}{7^7}$$

Example: 36 If a, b, c are positive real number then prove that $\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1+c}{2a}\ge\frac{5}{4}$

Solution: $\frac{a^3}{4b},\frac{b}{8c^2},\frac{1}{2a},\frac{c}{4a},\frac{c}{4a},$ Applying A.M. $\ge$ G.M.

$$\frac{\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{4a}+\frac{c}{4a}}{5}\ge\left(\frac{a^3}{4b}\cdot\frac{b}{8c^2}\cdot\frac{1}{2a}\cdot\left(\frac{c}{4a}\right)^2\right)^{1/5} \Rightarrow \frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{2a}\ge\frac{5}{4}$$

Example: 37 If m^{th} term of an H.P. is n, and n^{th} term is equal to m then prove that $(m + n)^{th}$ term is $\frac{mn}{m+n}$

Solution: m^{th} term of A.P. = $\frac{1}{n}$ $\Rightarrow$ $a + (m - 1)d = \frac{1}{n}$

n^{th} term of A.P. = $\frac{1}{m}$ $\Rightarrow$ $a + (n - 1)d = \frac{1}{m}$

$\Rightarrow$ $d = \frac{1}{mn}$, $a = \frac{1}{mn}$

$(m+n)^{th}$ term of A.P. $= a + (m + n - 1)d = \frac{1}{mn} + (m + n - 1)\frac{1}{mn} = \frac{m+n}{mn}$

$\therefore$ required $T_{m+n} = \frac{mn}{m+n}$.

Example: 38 If a, b, c are in HP, find the value of $\frac{b+a}{b-a}+\frac{b+c}{b-c}$.

Solution: a, b, c are in HP, then

$\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P.

$$S=\frac{b+a}{b-a}+\frac{b+c}{b-c}=\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}+\frac{\frac{1}{c}+\frac{1}{b}}{\frac{1}{c}-\frac{1}{b}}$$

Let $\frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c}$ = d

S = $\frac{\left(\frac{1}{a}+\frac{1}{b}\right)-\left(\frac{1}{c}+\frac{1}{b}\right)}{d}=\frac{\left(\frac{1}{a}-\frac{1}{c}\right)}{d}=\frac{2d}{d}=2$

Example: 39 If a^2, b^2, c^2 are in A.P. show that b + c, c + a, a + b are in H.P.

Solution: a^2, b^2, c^2 are in A.P.

Let b + c, c + a, a + b are in H.P.

Then $\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}$ are in A.P.

$\frac{2}{c+a}=\frac{1}{b+c}+\frac{1}{a+b}$

2(a + b) (b + c) = (2b + a + c) (a + c)

$2b^2 = a^2 + c^2$

Hence a^2, b^2, c^2 are in A.P.

So, if a^2, b^2, c^2 are in A.P. then b + c, c + a, a + b are in H.P.

Example: 40 If $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ and p, q, r are in A.P. then prove that x, y, z are in H.P.

Solution: Let $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ = k

p = $\frac{a-x}{kx}$, q = $\frac{a-y}{ky}$, r = $\frac{a-z}{kz}$

$2\left(\frac{a-y}{ky}\right)=\frac{a-x}{kx}+\frac{a-z}{kz}$

$2\left(\frac{a}{y}-1\right)=\frac{a}{x}-1+\frac{a}{z}-1$

$\frac{2a}{y}=\frac{a}{x}+\frac{a}{z}$, $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$

Hence x, y, z are in H.P.

Example: 41 If a, b, c, d be four distinct positive quantities in H.P., then show that

(a) a + d > b + c (b) ad > bc.

Solution: $\therefore$ a, b, c, d are in H.P.

(a) $\because$ A.M. > H.M.

For first three terms $\therefore \frac{a+c}{2}>b$ ...(i)

And for last three terms, $\frac{b+d}{2}>c$ ...(ii)

From (i) and (ii)

a + c + b + d > 2b + 2c

$\Rightarrow$ a + d > b + c

(b) Again G.M. > H.M.

For first three terms, $\sqrt{ac}>b$

$\Rightarrow$ $ac > b^2$ ...(i)

And for last three terms $\sqrt{bd} > c$

$\Rightarrow$ $bd > c^2$ ...(ii)

From (i) and (ii)

$(ac)(bd) > b^2c^2 \Rightarrow ad > bc.$

Example: 42 The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

Solution: Let a be the first term and d be the common difference of corresponding A.P.

$a + 6d = 10$

$a + 11d = 25$

$5d = 15$

$d = 3, a = -8$

$T_{20} = a + 19d = -8 + 19 \times 3 = 49$

20 term of H.P. = $\frac{1}{49}$

Example: 43 Insert 4 H.M between $\frac{3}{4}$ and $\frac{3}{19}$.

Solution: Let 'd' be the common difference of corresponding A.P..

so $d = \dfrac{\frac{19}{3} - \frac{4}{3}}{5} = 1.$

$\therefore \frac{1}{H_1} = \frac{4}{3} + 1 = \frac{7}{3}$ or $H_1 = \frac{3}{7}$

$\frac{1}{H_2} = \frac{4}{3} + 2 = \frac{10}{3}$ or $H_2 = \frac{3}{10}$

$\frac{1}{H_3} = \frac{4}{3} + 3 = \frac{13}{3}$ or $H_3 = \frac{3}{13}$

$\frac{1}{H_4} = \frac{4}{3} + 4 = \frac{16}{3}$ or $H_4 = \frac{3}{16}$.

Example: 44 The A.M. of two numbers exceeds the G.M. by 2 and the G.M. exceeds the H.M. by $\frac{8}{5}$; find the numbers.

Solution: Let the numbers be a and b, now using the relation

$G^2 = AH = (G + 2)\left(G - \frac{8}{5}\right) \Rightarrow G = 8$ and $A = 10$

i.e. $ab = 64$

also

$a + b = 20$

Hence the two numbers are 4 and 16.

Example: 45 If $n > 0$, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution: Using the relation A.M. ≥ G.M. on the numbers 1, 2, 2^2, 2^3,..........., 2^{n-1} , we have

$$\frac{1+2+2^2+.......+2^{n-1}}{n} > (1.2.\ 2^2.\ 2^3.\ ....\ .2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \frac{2^n-1}{2-1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \quad \Rightarrow \quad 2^n - 1 > n\ 2^{\frac{(n-1)}{2}}$$

$$\Rightarrow 2^n > 1 + n\,2^{\frac{(n-1)}{2}} \quad \Rightarrow 2^n > 1 + n.\ \sqrt{2^{n-1}}$$

---

9. ARITHMETIC GEOMETRIC PROGRESSION (A.G.P.) :

If each term of a Progression is the product of the corresponding terms of an A.P. and a G.P., then it is called arithmetic- geometric progression (A. G.P.)

Ex. a, (a + d) r, (a + 2d) r^2 , ...

The general term (n^{th} term) of an A.G.P. is

$T_n = [a + (n-1) d]\ r^{n-1}$

To find the sum of n terms of an A.G.P. we suppose its sum S, multiply both sides by the common ratio of the corresponding G.P. and then subtract as in following way and we get a G.P. whose sum can be easily obtained.

$S_n = a + (a + d) r + (a + 2d) r^2 + ...[a + (n-1) d] r^{n-1}$

$rS_n = ar + (a+d) r^2 + ...+ [a+(n-1) d] r^n$

After subtraction we get

$S_n (1-r) = a+ r.d + r^2 d +...dr^{n-1} - [a + (n-1)d] r^n$

After solving

$$S_n = \frac{a}{1-r} + \frac{r.d(1-r^{n-1})}{(1-r)^2} - \frac{\left[a+(n-1)d\right]r^n}{1-r}$$

and $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$, [if 0 < |r| < 1]

Note : This is not a standard formula. This is only to understand the procedure for finding the sum of an A.G.P. However formula for sum of infinite terms can be used directly.

10. MISCELLANEOUS SEQUENCES

10.1 Type-1 :

Sequences dealing with $\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$

(1) $\sum n = \frac{n(n+1)}{2}$

(2) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left(\sum n\right)^2$

DETECTIVE MIND

- $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$
- $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$
- $\sum_{r=1}^{n} k = k\sum_{r=1}^{n} 1 = kn$

Note : (i) Sum of first n odd natural numbers

$\Rightarrow \quad \sum_{r=1}^{n}(2r-1) = n^2$

(ii) If r^{th} term of an A.P.

$T_r = Ar^3 + Br^2 + Cr + D$, then

sum of n term of AP is

$S_n = \sum_{r=1}^{n} T_r = A\sum_{r=1}^{n} r^3 + B\sum_{r=1}^{n} r^2 + C\sum_{r=1}^{n} r + D\sum_{r=1}^{n} 1$

Example: Find the sum of the series to n terms whose n^{th} term is 3n + 2.

Solution: $S_n = \Sigma T_n = \sum(3n + 2) = 3\Sigma n + \Sigma 2 = \frac{3(n+1)\ n}{2} + 2n = \frac{n}{2}(3n + 7)$

10.2 Type-2 (Using method of difference):

If T_1, T_2, T_3,……… are the terms of a sequence then the terms

$T_2 - T_1, T_3 - T_2, T_4 - T_3$ ………..

Some times this is in A.P. and some times in G.P. For such series we first compute their n^{th} term and then compute the sum to n terms, using sigma notation.

Example: Find the sum of series.

6 + 13 + 22 + 33 + ……………n terms

Solution: Let $S = 6 + 13 + 22 + 33 + \ldots\ldots\ldots + T_n$

$S = \quad 6 + 13 + 22 + \ldots\ldots\ldots + T_{n-1} + T_n \}$

$0 = \quad 6 + 7 + 9 + 11 + \ldots\ldots\ldots - T_n$

or $T_n = 6 + (7 + 9 + 11 + \ldots\ldots)$

$= 6 + \left[\frac{n-1}{2}\left[2\times7+(n-2)2\right]\right] = 6 + \left[\frac{(n-1)}{2}(14+2n-4)\right]$

$= 6 + (n-1)(n+5) = 6 + n^2 + 4n - 5$

$T_n = n^2 + 4n + 1$

$\therefore \quad S_n = \sum_{n=1}^{n} T_n = \frac{n(n+1)(2n+1)}{6} + \frac{4(n)(n+1)}{2} + n$

$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+4\right] + n = \frac{n(n+1)}{2}\left[\frac{2n+13}{3}\right] + n = \frac{n(n+1)(2n+13)}{6} + n$

10.3 TYPE – 3 (Splitting the n^{th} term as a difference of two) :

Here is a series in which each term is composed of the reciprocal of the product of r factors in A.P., the first factor of the several terms are in A.P.

Example: Find the sum of n terms of the series $\frac{3}{1\cdot2\cdot4} + \frac{4}{2\cdot3\cdot5} + \frac{5}{3\cdot4\cdot6} + \ldots\ldots$ and also find sum of infinite terms (S_∞)

Solution: $\frac{3}{1\cdot2\cdot4} + \frac{4}{2\cdot3\cdot5} + \frac{5}{3\cdot4\cdot6} + \ldots\ldots \frac{(n+2)}{n(n+1)(n+3)}$

$= \frac{3^2}{1\cdot2\cdot3\cdot4} + \frac{4^2}{2\cdot3\cdot4\cdot5} + \ldots\ldots + \frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$

$T_n = \frac{n^2+4n+4}{n(n+1)(n+2)(n+3)} = \frac{n^2+4n+3}{n(n+1)(n+2)(n+3)} + \frac{1}{n(n+1)(n+2)(n+3)}$

$T_n = \frac{1}{n(n+2)} + \frac{1}{n(n+1)(n+2)(n+3)}$

Now, $S_1 = \sum_{n=1}^{n}\frac{1}{n(n+2)}$, $S_2 = \sum_{n=1}^{n}\frac{1}{n(n+1)(n+2)(n+3)}$

$S_1 = \frac{1}{1\cdot3} + \frac{1}{2\cdot4} + \frac{1}{3\cdot5} + \ldots\ldots \frac{1}{n(n+2)}$

$S_1 = \frac{1}{2}\left[\frac{3-1}{1\cdot3} + \frac{4-2}{2\cdot4} + \frac{5-3}{3\cdot5} + \ldots\ldots + \frac{(n+2)-n}{n(n+2)}\right]$

$= \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \ldots\ldots + \frac{1}{n} - \frac{1}{n+2}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$

$S_2 = \frac{1}{3}\left[\frac{1}{1\cdot2\cdot3} - \frac{1}{(n+1)(n+2)(n+3)}\right]$ $\quad S = S_1 + S_2.$

10.4 TYPE-4:

Here is a series in which each terms is composed of r factor in A.P., the first factor of the several terms are in A.P.

Example: $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots\ldots\ldots\ldots$ up to n terms

Solution: $T_n = n(n + 1)(n + 2)(n + 3)$

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$T_1 = \frac{1\cdot2\cdot3\cdot4\cdot5}{5} - 0$$

$$T_2 = \frac{2\cdot3\cdot4\cdot5\cdot6}{5} - \frac{1\cdot2\cdot3\cdot4\cdot5}{5}$$

$$T_3 = \frac{3\cdot4\cdot5\cdot6\cdot7}{5} - \frac{2\cdot3\cdot4\cdot5\cdot6}{5}$$

…… ………… …… ……

…… ………… …… ……

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

SOLVED EXAMPLES

Example: 46 (a) If $9 + 99 + 999 + \ldots\ldots +$ upto 49 terms $= 10\frac{(10^{\lambda} - 1)}{\mu} - 49$, where $\lambda, \mu \in N$ then find the value of $\lambda + \mu$.

(b) $0.9 + 0.99 + 0.999 + \ldots\ldots$ up to 51 terms $= 51 - \frac{1}{p}\left(1 - \frac{1}{10^q}\right)$ where $p, q \in N$ then find the value of $p + q$.

Solution: (a) $S = 9 + 99 + 999 + \ldots\ldots +$ upto 49 terms

$S = 10 - 1 + 10^2 - 1 + 10^3 - 1 + \ldots\ldots + 10^{49} - 1$

$= (10 + 10^2 + 10^3 + \ldots\ldots 10^{49}) - 49$

$$S = 10\cdot\left(\frac{10^{49} - 1}{9}\right) - 49$$

$\lambda + \mu = 49 + 9 = 58$

(b) $S = 0.9 + 0.99 + 0.999 + \ldots\ldots$ up to 51 terms

$$= \frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots.. \text{ up to 51 terms}$$

$$= 1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \ldots\ldots + 1 - \frac{1}{10^{51}}$$

$$= 51 - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\ldots + \frac{1}{10^{51}}\right)$$

$$= 51 - \frac{\frac{1}{10}\left(1-\frac{1}{10^{51}}\right)}{1-\frac{1}{10}} = 51 - \frac{1}{9}\left(1-\frac{1}{10^{51}}\right)$$

$\therefore$ p + q = 60

Example: 47 Find the sum of series $\frac{3}{19}+\frac{33}{19^2}+\frac{333}{19^3}+\frac{3333}{19^4}+......\infty$

Solution: $S = \frac{3}{19}+\frac{33}{19^2}+\frac{333}{19^3}+\frac{3333}{19^4}+......\infty$

or $S = \frac{3}{9}\left[\frac{9}{19}+\frac{99}{19^2}+\frac{999}{19^3}+......\infty\right]$

$$\Rightarrow \frac{3}{9}\left[\frac{10-1}{19}+\frac{10^2-1}{19^2}+\frac{10^3-1}{19^3}......\infty\right]$$

$$\Rightarrow \frac{3}{9}\left[\left(\left(\frac{10}{19}\right)+\left(\frac{10}{19}\right)^2+\left(\frac{10}{19}\right)^3+......\infty\right)-\left(\frac{1}{19}+\frac{1}{19^2}+......\infty\right)\right]$$

$$S = \frac{3}{9}\left[\frac{10/19}{1-10/19}-\left(\frac{1/19}{1-1/19}\right)\right]$$

$$S = \frac{3}{9}\left[\frac{19}{18}\right] \Rightarrow \frac{19}{54}$$

Example: 48 If the sum to infinity of the series 3 + (3 + d)$\frac{1}{4}$ + (3 + 2d)$\frac{1}{4^2}$ + ... ∞ is $\frac{44}{9}$, then find d.

Solution: $S = 3+(3+d)\frac{1}{4}+(3+2d)\frac{1}{4^2}+......\infty$

$$\Rightarrow \frac{1}{4}S = (3)\frac{1}{4}+(3+d)\frac{1}{4^2}+......\infty$$

Subtracting (2) from (1), we have

$$\frac{3}{4}S = 3+(d)\frac{1}{4}+(d)\frac{1}{4^2}+......\infty$$

$$= 3+\frac{\frac{d}{4}}{1-\frac{1}{4}} = 3+\frac{d}{3} \Rightarrow S = 4+\frac{4d}{9}$$

Given,

$$4+\frac{4d}{9} = \frac{44}{9}$$

$$\Rightarrow \frac{4d}{9} = \frac{8}{9} \Rightarrow d = 2$$

Example: 49 If $|x| < 1$ then compute the sum

(a) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots \infty$

(b) $1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots \infty$

Solution: (a) $S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$ ________(1)

$xS = x + 2x^2 + 3x^3 + \ldots\ldots\ldots\ldots\infty$

$S(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots$

$$S(1-x) = \frac{1}{1-x}$$

$$S = \frac{1}{(1-x)^2}$$

(b) $S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\infty$

$xS = \quad x + 3x^2 + 6x^3 + \ldots\ldots\ldots\ldots\infty$

$(1 - x)S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$

Or by result of equation (1)

$$(1-x)S = \frac{1}{(1-x)^2}$$

$$S = \frac{1}{(1-x)^3}$$

Example: 50 Compute the sum $(31)^2 + (32)^2 + (33)^2 + \ldots\ldots + (50)^2$

Solution: $S = (31^2) + (32^2) + (33^2) + \ldots\ldots + (r + 30)^2 + \ldots\ldots (50)^2$

$$S = \sum_{r=1}^{20}(r+30)^2 = \sum_{r=1}^{20} r^2 + 60r + 900$$

$$\Rightarrow \frac{20\times21\times41}{6} + \frac{60\times20\times21}{2} + 900\times20$$

$\Rightarrow$ 33470

Example: 51 Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots\ldots$

Solution: $T_n = n(n + 1)(n + 2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n}\left(k^3 + 3k^2 + 2k\right)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}\{n^2+n+4n+2+4\} = \frac{n(n+1)}{4}(n^2+5n+6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Example: 52 The sum to n terms of the series $1 + 5\left(\frac{4n+1}{4n-3}\right) + 9\left(\frac{4n+1}{4n-3}\right)^2 + 13\left(\frac{4n+1}{4n-3}\right)^3 + ...$ is .

Solution: Let $x = \frac{4n+1}{4n-3}$, then

$1 - x = \frac{-4}{4n-3}, \frac{1}{1-x} = -\frac{(4n-3)}{4}$

$\frac{x}{1-x} = -\frac{(4n+1)}{4}$

$S = 1 + 5x + 9x^2 + ... + (4n-3)x^{n-1}$

$Sx = x + 5x^2 + ... (4n-3)x^n$

$S - Sx = 1 + 4x + 4x^2 + ... + 4x^{n-1} - (4n-3)x^n.$

$S(1-x) = 1 + \frac{4x}{1-x}[1 - x^{n-1}] - (4n-3)x^n$

$S = \frac{1}{1-x}\left[1 + \frac{4x}{1-x} - \frac{4x^n}{1-x} - (4n-3)x^n\right] = -\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n] = n(4n-3).$

Example: 53 Find the sum to n terms of the series 5, 7, 13, 31, 85 + ...

Solution: Successive difference of terms are in G.P. with common ratio 3.

$T_n = a(3)^{n-1} + b$

$\Rightarrow a + b = 5$

and $3a + b = 7 \Rightarrow a = 1, b = 4$

$\therefore T_n = 3^{n-1} + 4$

$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = (1 + 3 + 3^2 + ... + 3^{n-1}) + 4n$

$\frac{1}{2}[3^n + 8n - 1]$

Example: 54 Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 + ...

Solution: $T_r = (3r-1)(3r+2) = 9r^2 + 3r - 2$

$S_n = \sum_{r=1}^{n} T_r = 9\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$

$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 3\left(\frac{n(n+1)}{2}\right) - 2n = 3n(n+1)^2 - 2n$

Example: 55 Sum to n terms of the series $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+$...........

Solution: $T_n = \frac{1}{(3n-2)(3n+1)(3n+4)}$

$= \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$

$= \frac{1}{6}\left[\left(\frac{1}{1.4}-\frac{1}{4.7}\right)+\left(\frac{1}{4.7}-\frac{1}{7.10}\right)+......+\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$

$= \frac{1}{6}\left[\frac{1}{4}-\frac{1}{(3n+1)(3n+4)}\right]$

Example: 56 Find the general term and sum of n terms of the series

1 + 5 + 19 + 49 + 101 + 181 + 295 + ...

Solution: The sequence of difference between successive term 4, 14, 30, 52, 80 .....

The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P.

so let nth term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ ....(i)

$8a + 4b + 2c + d = 5$ ....(ii)

$27a + 9b + 3c + d = 19$ ....(iii)

$64a + 16b + 4c + d = 49$ ....(iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \Rightarrow$ $T_n = n^3 - n^2 + 1$

$s_n = \Sigma(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$