Maths Optional

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(2) Convergent series may be added or subtracted term by term.

i.e., if $\sum u_n = u$ and $\sum v_n = v$ then

$$\sum (u_n \pm v_n) = u \pm v.$$

Note (1) The behaviour of a series as regards convergence is not altered by

(i) the alteration, omission or addition of a finite number of terms; or

(ii) multiplication of all the terms by a non-zero number.

+ve term series

$\sum u_n$ is a +ve term series.

$u_n > 0 \quad \forall n$

Let $\langle s_n \rangle$ be the S.O.P.S.

$s_n = u_1 + u_2 + \cdots + u_n$

$s_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$

$s_n - s_{n-1} = u_n > 0$

$\implies s_n > s_{n-1} \quad \forall n.$

Result: If a series $\sum u_n$ converges to the sum 'u' then so does any series obtained from $\sum u_n$ by grouping the terms in brackets without altering the order of the terms. Converse is not true.

$\therefore$ $\sum u_n$ is also divergent.

<u>Note</u>: The s.o.p.s of a series with negative terms is monotonically decreasing and hence a series with negative terms converges iff its s.o.p.s is bdd below.

$\therefore$ s.o.p.s is mono $\uparrow$

$\langle s_n \rangle$ is cgt when it is bdd above.

$\therefore$ Consequently, $\sum u_n$ converge.

Also if $\langle s_n \rangle$ is not bdd above then it will diverge to $+\infty$.

Note: ① $\sum u_n$ — positive term series.

terms monotonically decreasing.

$\lim n u_n \neq 0$

$\implies \sum u_n$ div. to $+\infty$.

Ex: $\sum \frac{1}{n}$

let $u_n = \frac{1}{n}$

$\lim n u_n = \lim n \cdot \frac{1}{n} = 1 \neq 0$

$\therefore \sum \frac{1}{n}$ diverges.

If S.O.P.S is not bdd. below then series diverges to $-\infty$.

Result: If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $n u_n \to 0$.

But $\sum \frac{1}{n \log n}$ is not cgt.
(divergent)

Geometric series: The positive term geometric series

$1 + r + r^2 + r^3 + \cdots$ converges for $r < 1$ and diverges to $+\infty$ for $r \geq 1$.

② The condition $n u_n \to 0$ is only necessary not sufficient.

Consider the ex.

$$\sum \frac{1}{n \log n}$$

Let $u_n = \frac{1}{n \log n}$

$$\lim n u_n = \lim_{n \to \infty} \frac{1}{\log n} = 0$$

$\therefore s_n < \frac{1}{1-r} \quad \forall n$

$\therefore \langle s_n \rangle$ is bdd above.

$\therefore$ cgt is cgt.

$\therefore$ The series

$1 + r + r^2 + \cdots$ is

cgt.

Case (ii) $r = 1$

$\langle s_n \rangle$ be the s.o.p.s.

$s_n = 1 + 1 + 1 + \cdots n$ times

$= n$

Proof: Case (i) $0 < r < 1$

Let $\langle s_n \rangle$ be the s.o.p.s.

$s_n = 1 + r + r^2 + \cdots + r^{n-1}$

$= \frac{1-r^n}{1-r}$

$= \frac{1}{1-r} - \frac{r^n}{1-r} < \frac{1}{1-r}$

$\therefore \langle s_n \rangle$ is not bdd above.

$\therefore \langle s_n \rangle \to +\infty$.

$\therefore 1 + r + r^2 + \cdots$ —. diverges to $+\infty$.

<u>A Comparison series:</u>

A +ve term series $\sum \frac{1}{n^p}$ is cgt if $p > 1$.

<u>Note</u>: It diverges if $0 < p \le 1$.

$\langle s_n \rangle$ is not bdd above.

$\therefore \langle s_n \rangle \to +\infty$.

$\therefore 1 + r + r^2 + \cdots$ — diverges for $r = 1$.

<u>Case III</u> <u>$r > 1$</u>

$\langle s_n \rangle$ be the S.O.P.S.

$$s_n = 1 + r + r^2 + \cdots + r^{n-1}$$
$$> 1 + 1 + 1 + \cdots + 1$$

$\therefore s_n > n$ then.

Second comparison test

If $\sum u_n$ and $\sum v_n$ are two +ve term series such that

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \quad \forall n \geq m$$

(m, a fixed +ve integer)

then (I) $\sum v_n$ converges $\Rightarrow$ $\sum u_n$ converges.

(II) $\sum u_n$ div. $\Rightarrow$ $\sum v_n$ div.

First comparison test

Let $\sum u_n$ and $\sum v_n$ be the two +ve term series s.t.

$$u_n \leq K v_n \quad \forall n \geq m$$

(K, a fixed +ve number and m, a fixed +ve integer)

(I) $\sum v_n$ converges $\Rightarrow$ $\sum u_n$ converges.

(II) $\sum u_n$ diverges $\Rightarrow$ $\sum v_n$ diverges.

estimate the magnitude of the nth term.

$$u_n \sim \underline{v_n}$$

For large values of n

(i) $$\frac{1}{\sqrt{n+1}+\sqrt{n-1}} \sim \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}} \sim \frac{1}{\sqrt{n}}$$

(ii) $$\sqrt{n^3+1} \sim (n^3)^{\frac{1}{2}} = n^{\frac{3}{2}}$$

<u>Limit form test:</u> Let $\sum u_n$ & $\sum v_n$ are two +ve term series such that $\lim \frac{u_n}{v_n} = l$ (a non-zero finite) then $\sum u_n$ and $\sum v_n$ will converge or diverge together.

<u>Note (i)</u>: For a successful application of comparison test we

Prob: Show that the series

$$1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots$$

is convergent.

$\rightarrow$

$$1=1$$

$$\frac{1}{\lfloor 2}=\frac{1}{2}$$

$$\frac{1}{\lfloor 3}<\frac{1}{2^2}$$

$$\frac{1}{\lfloor 4}<\frac{1}{2^3}$$

$$\vdots \quad \vdots \quad \vdots$$

$\left[\lfloor n > 2^{n-1} \quad \forall n>2\right]$

(iii) $\frac{n^r}{(1+n)^s} \sim \frac{n^r}{n^s} = n^{r-s}$

(iv) $\sin\frac{1}{n} \sim \frac{1}{n}$

Note (2):

$$e^{an} > n^b > (\log n)^c$$

a, b, c are +ve number.

prob: Show that the series

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \cdots$$

diverges for $p > 0$.

$\to \quad n > (\log n)^p$, for $p > 0$, $n > 1$

$\Rightarrow \frac{1}{n} < \frac{1}{(\log n)^p} \quad \forall n > 1$

$\therefore \sum \frac{1}{n}$ is divergent.

$\therefore 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots < \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots}_{\text{cgt}}$

$\therefore \sum \frac{1}{\lfloor n}$ is also cgt.

$\longrightarrow \quad u_n = \frac{1}{\lfloor n} \quad$ let $v_n = \frac{1}{2^{n-1}}$

$u_n < v_n \quad \forall n > 2$

$\sum v_n$ converge

$\therefore \sum u_n$ " .

$$\rightarrow \quad u_n = \frac{(2n-1)\times 2n}{(2n+1)^2\cdot(2n+2)^2} \sim \frac{2n\times 2n}{(2n)^2(2n)^2}$$

$$= \frac{1}{4n^2}$$

Let $v_n = \frac{1}{n^2}$ $\qquad \sim \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{(2n-1)\times 2n}{(2n+1)^2\cdot(2n+2)^2} \times n^2$$

$$= \frac{\cancel{n^4}\left[\left(2-\frac{1}{n}\right)\times 2\right]}{\cancel{n^4}\left[\left(2+\frac{1}{n}\right)^2\cdot\left(2+\frac{3}{n}\right)^2\right]}$$

$\therefore$ By comparison test

$\sum \frac{1}{(\log n)^p}$ also diverges

for $p>0$

$n>1$.

Prob: Show that the series

$$\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\cdots$$

is cgt.

Prob: investigate the behaviour of $\sum \sin\frac{1}{n}$.

Sol^n: $\sin\frac{1}{n} \sim \frac{1}{n}$

let $u_n = \sin\frac{1}{n}$

$v_n = \frac{1}{n}$

$$\lim \frac{u_n}{v_n} = \lim \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$$

∴ By Limit form comparison test,

$$\lim \frac{u_n}{v_n} = \frac{1}{4} \neq 0$$

Non-zero finite

∴ $\sum u_n$ and $\sum v_n$ will have same nature.

∴ $\sum v_n = \sum \frac{1}{n^2}$ converge.

∴ $\sum u_n$ also converge.

$\to$ Let $u_n = (n^3+1)^{1/3} - n$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$

$$= n\left[\cancel{1} + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\left(\frac{1}{n^3}\right)^2 + \cdots - \cancel{1}\right]$$

$$= \frac{1}{3n^2} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2} \cdot \frac{1}{n^5} + \cdots$$

$$\sim \frac{1}{n^2}$$

Σu_n and Σv_n will have same nature.

$\therefore \Sigma v_n$ diverges.

$\therefore \Sigma u_n$ also diverges.

Prob: Investigate the behaviour of the series whose n^{th} term is $\left\{(n^3+1)^{\frac{1}{3}} - n\right\}$

$\therefore \sum \frac{1}{n^2}$ converges.

$\therefore \sum u_n$ also converges.

<u>Prob</u>: Test the convergence of the series $\sum \frac{1}{n^{1+\frac{1}{n}}}$

$\rightarrow$ Let $u_n = \frac{1}{n^{1+\frac{1}{n}}} \sim \frac{1}{n}$

Let $v_n = \frac{1}{n}$

Let $v_n = \frac{1}{n^2}$

$\therefore \lim \frac{u_n}{v_n} = \lim \left[\frac{1}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{\lfloor 2} \cdot \frac{1}{n^3} \cdots \right]$

$= \frac{1}{3} \neq 0$

$\therefore$ By limit form of comparison test.

$\sum u_n$ and $\sum v_n$ will have the same nature.

$\sum u_n$ and $\sum v_n$ will have same nature.

$\because \sum v_n = \sum \frac{1}{n}$ diverge.

$\therefore \sum u_n$ also diverge.

$$\frac{u_n}{v_n} = \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}}$$

$$= \frac{1}{n^{\frac{1}{n}}}$$

$$\therefore \lim \frac{u_n}{v_n} = \lim \frac{1}{n^{\frac{1}{n}}}$$

$$= \frac{1}{1} = 1 \neq 0$$

$\therefore$ By limit form of comparison test.