KGS IAS

Maths Optional

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(11) $\mathbb{Q}$

Every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

$p \in \mathbb{R}$.

$]p-\varepsilon, p+\varepsilon[\cap \mathbb{Q}$ will contain infinitely many rational number.

For each $\varepsilon > 0$

Ex: (1) Every point of the set $\mathbb{R}$ is the limit point of $\mathbb{R}$.

$p \in \mathbb{R}$

$]p-\varepsilon, p+\varepsilon[\cap \mathbb{R} =$ infinite set

for each $\varepsilon > 0$

(v) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

0 is a limit point of S.

$]-\varepsilon, \varepsilon[\cap S =$ infinite set.

(vi) $S = \left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}$

$= \{1 + \frac{1}{n} : n \in \mathbb{N}\}$.

$(1-\varepsilon, 1+\varepsilon) \cap S =$ infinite set

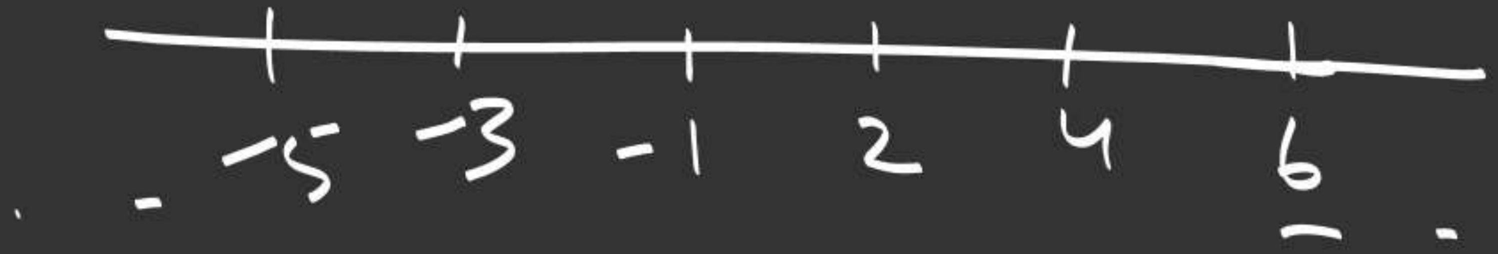
for each $\varepsilon > 0$

(iii) $\mathbb{R} \sim \mathbb{Q}$ — set of irrational.

Any point of $\mathbb{R}$ is a limit point of $\mathbb{R} \sim \mathbb{Q}$.

(iv) $S =]a, b[$ or $[a, b]$ or $]a, b]$ or $[a, b[$

Every point of S is a limit point of S.



$$\lim_{n\to\infty} (-1)^n \cdot n = \begin{cases} -\infty, & n \text{ is odd} \\ +\infty, & n \text{ is even} \end{cases}$$

S has no limit point.

$\therefore$ 1 is a limit point.

$$\lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$= 1 + 0 = 1$$

(VII) $S = \{(-1)^n \cdot n : n \in \mathbb{N}\}$

$S = \{-1, 2, -3, 4, -5, \ldots\}$

Derived set: The set of all limit points of the set $S \subseteq \mathbb{R}$ is called the derived set of S.

Denoted by S' or $D(S)$.

i.e., $S' = \{x \in \mathbb{R} : x \text{ is a limit pt. of } S\}$

Result: If the supremum of a set doesn't belong to the set, then it is the limit point of the set.

Result: If the infimum of a set doesn't belong to the set, then it is the limit point of the set.

Adherent point : A real no. p is called an adherent point of a set $S \subseteq \mathbb{R}$ if every nbd. of p contains a point of S

i.e. If N is a nbd of p

then $N \cap S \neq \emptyset$.

Ex (i) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$D(S) = \{0\}$.

(ii) $S = \mathbb{R}$

$D(S) = \mathbb{R}$

(iii) $S = \emptyset$

$D(\emptyset) = \mathbb{R}$.

Ex: $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$0 \notin S$, 0 is a l. pt. of S.

0 is also an adherent pt. of S.

$1 \in S$, 1 is an adherent pt. of S but 1 is not a limit point of S.

Note: ① A real number p is an adherent pt. of S $\iff$ either $p \in S$ or $p \in D(S)$.

② Every limit pt. of a set S is an adherent pt. but converse is not true.

Ex (i) ϕ

$$cl\,\phi = \phi \cup D(\phi)$$
$$= \phi \cup \mathbb{R}$$
$$= \mathbb{R}.$$

(ii) $S = \mathbb{R} \sim \phi.$

$$cl\,S = \mathbb{R}.$$

Closure of a set: The set of all adherent points of a set S is called closure of the set S.

Denoted by $\tilde{S}$ or $cl\,S$.

i.e. $\tilde{S} = S \cup D(S)$

(V) $cl\,\mathbb{R} = \mathbb{R} \cup D(\mathbb{R})$

$= \mathbb{R} \cup \mathbb{R}$

$= \mathbb{R}$

(VI) $S = (a, b)$

$cl\,S = S \cup D(S)$

$= (a, b) \cup [a, b]$

$= [a, b].$

(III) $\mathbb{N}$

$cl\,\mathbb{N} = \mathbb{N} \cup D(\mathbb{N})$

$= \mathbb{N} \cup \phi$

$= \mathbb{N}.$

(IV) $cl\,(\phi) = \phi \cup D(\phi)$

$= \phi \cup \phi$

$= \phi$

Existence of limit point

Bolzano-Weierstrass th: (B W Th)

Every infinite bounded set has a limit point.

Ex: $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

$0 < x \leq 1, \forall x \in S$

S is bdd & infinite.

$D(S) = \{0\}$.

$cl[a,b] = [a,b]$

$cl(]a,b]) = [a,b]$

$cl([a,b[) = [a,b]$

VII $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

$cl\,S = S \cup D(S)$

$= \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

(2) $A \subseteq B \Rightarrow D(A) \subseteq D(B)$

$p \in D(A)$

$\Rightarrow p$ is a limit pt of A

$\Rightarrow$ For every nbd N of 'p'

$N \cap A \sim \{p\} \neq \phi$

$\because A \subseteq B$

$\therefore N \cap B \sim \{p\} \neq \phi$

$\therefore p$ is a l-pt of B

$\Rightarrow p \in D(B)$.

Note: Converse of the above th. is not true.

$\mathbb{R}$ — infinite but not bdd.

$D(\mathbb{R}) = \mathbb{R}$.

Some Results on Derived set

(1) If A and B are two subsets of $\mathbb{R}$, then

Denoted by $\overline{\lim} S$

The smallest of all the limit points of a set S is called inferior limit (limit infirior) of the set.

Denoted by $\underline{\lim} S$.

(b) $D(A \cup B) = D(A) \cup D(B)$

(c) $D(A \cap B) \subseteq D(A) \cap D(B)$.

(2) The derived set of any bounded set is also bdd

(3) The greatest of all the limit points of a set S is called limit superior of the set.

<u>Note</u> ① The supremum (or infimum) of a bounded set S is always a member of $\mathrm{cl}\,S$.

② If S is bdd then $\mathrm{cl}\,S$ is also bdd.

<u>Ex:</u> $S =]0,1[$

$D(S) = [0,1]$

$\overline{\lim}\, S = 1 \quad \left(\underline{\lim}\, S \le \overline{\lim}\, S\right)$

$\underline{\lim}\, S = 0$

<u>Ex:</u> $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

$D(S) = \{0\}$.

$\overline{\lim}\, S = \underline{\lim}\, S = 0$

Note: S is closed $\Leftrightarrow$ $\text{cl}\, S = S$

Ex: ① $S = \{2\}$

$S^c = \mathbb{R} \sim S = \,]-\infty, 2[\, \cup \,]2, \infty[$

open

S is closed.

Closed set: A set $S \subseteq \mathbb{R}$ is a closed set iff $\mathbb{R} \sim S$ is an open set.

$\rightarrow$ A set $S \subseteq \mathbb{R}$ is called a closed set iff it contains all its limit points.

i.e. $D(S) \subseteq S$.

(iii) $\mathbb{N}, \mathbb{Z}$ — closed.

$cl\,\mathbb{N} = \mathbb{N}$

$cl\,\mathbb{Z} = \mathbb{Z}$.

(iv) ϕ — Not closed.

$D(\phi) = \mathbb{R}$

$cl\,\phi = \phi \cup D(\phi) = \mathbb{R}$

$cl\,\phi \neq \phi$.

(ii) S is a non-empty finite set.

$D(S) = \phi$

$$\begin{aligned} cl\,S &= S \cup D(S) \\ &= S \cup \phi \\ &= S. \end{aligned}$$

$\therefore$ S is a closed set.

$\mathbb{R}$ — closed as well as open.

VIII ϕ — closed

$cl\,\phi = \phi$

ϕ — open set.

V $\mathbb{R} \sim \mathbb{Q}$ — NOT closed.

VI $S =]a, b]$ or $[a, b[$ or $]a, b[$ or $[a, b]$ → closed set (only)

$cl\,S = [a, b]$

VII $\mathbb{R}$ — closed

$cl\,\mathbb{R} = \mathbb{R}$

To prove: F is closed.

$F^c = \mathbb{R} \sim F$

$= \left(\bigcap_{\lambda \in \Delta} F_\lambda \right)^c$

$= \bigcup_{\lambda \in \Delta} F_\lambda^c$ [By De-Morgan's rule] ①

$\because$ Each F_λ is closed.

$\therefore$ Each F_λ^c is open.

Th: The intersection of an arbitrary family of closed set is closed.

Proof: Let $\mathcal{F} = \{ F_\lambda : \lambda \in \Delta,$ Δ - index set $\}$ be an arbitrary family of closed set.

Let $F = \bigcap_{\lambda \in \Delta} F_\lambda$

Theorem: The union of a finite collection of closed set is closed.

Proof: Let $F_1, F_2, \ldots F_n$ be closed set.

Let $F = F_1 \cup F_2 \cup \cdots \cup F_n$

$F^c = F_1^c \cap F_2^c \cap \cdots \cap F_n^c$ — (1)

$\because$ Each F_i is a closed set

We know that union of an arbitrary family of open set is open.

$\therefore$ (1) $\Rightarrow \bigcup_{\lambda \in \Delta} F_\lambda^c$ is open

$\therefore$ F^c is open

$\therefore$ F is a closed set.

Note: The union of infinite collection of closed set may not be closed.

Ex: $F_n = \left[\frac{1}{n}, 2\right]$, $n \in \mathbb{N}$.

$\therefore$ Each F_n is closed.

$F_1 \cup F_2 \cup F_3 \cup - - -$

$= [1, 2] \cup \left[\frac{1}{2}, 2\right] \cup \left[\frac{1}{3}, 3\right] \cup - - -$

$=]0, 2]$ — not closed.

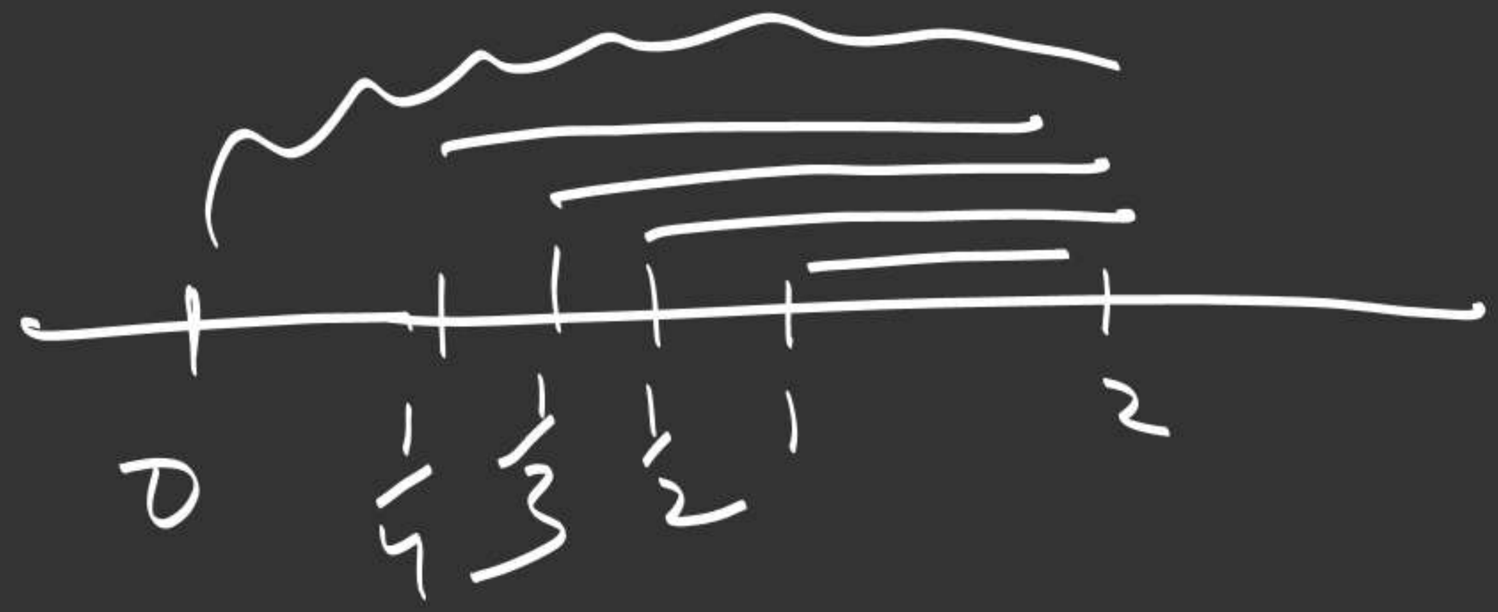
$\therefore$ Each F_i^c is an open set.

We know that intersection of finitely many open set is open.

$\therefore$ (1) $\Rightarrow$

F^c is open

$\therefore$ F is closed.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