KGS IAS

Maths Optional

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Th: Every convergent sequence is bounded. Converse is not true.

Proof: Let $\langle a_n \rangle$ be a convergent seq. And let $\lim a_n = l$.

Choose $\varepsilon = 1$,

$\because \lim a_n = l$

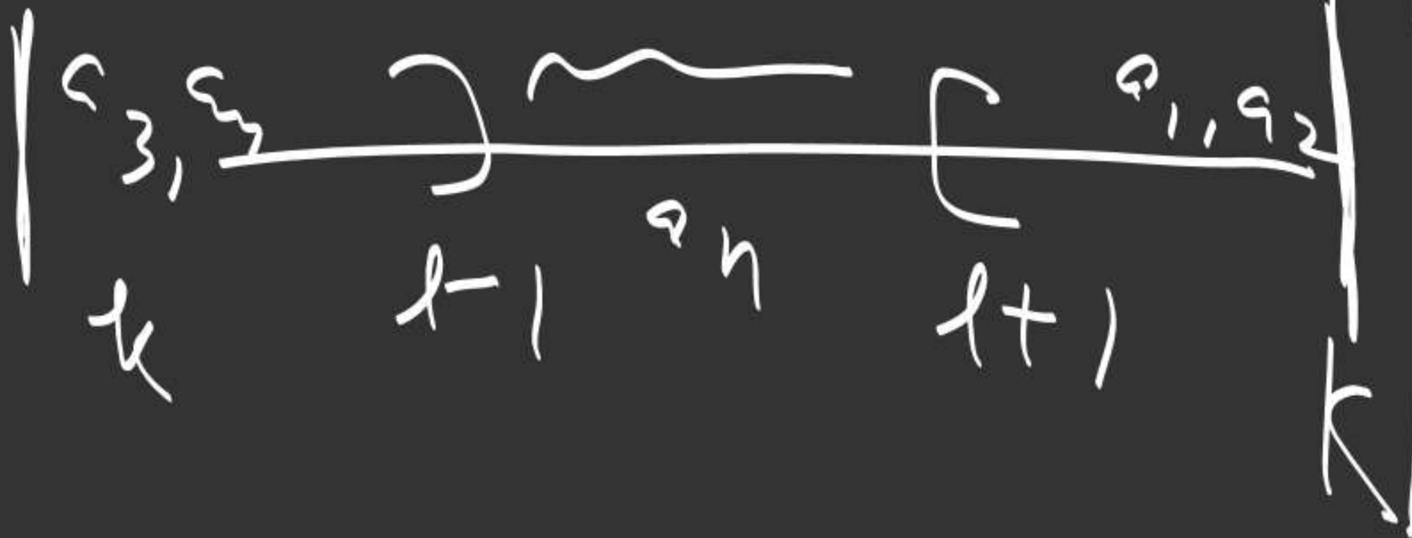
$\therefore$ corr. to $\varepsilon = 1$, $\exists$ a +ve integer m s.t.

$$|a_n - l| < 1 \quad \forall n \geq m$$

$$\Rightarrow \quad l-1 < a_n < l+1 \quad \forall n \geq m$$

Let $K = \max\{a_1, a_2, \ldots a_{m-1}, l+1\}$

$\therefore a_n \leq K \quad \forall n$ — (1)



Consider the example

$\langle (-1)^n \rangle$

$\equiv \langle -1, 1, -1, 1 - - - \rangle$

It is bdd.

It is not cgt.

oscillates finitely.

Let $k = \min\{a_1, a_2, \ldots a_{m-1}, l-1\}$

$\therefore k \le a_n \quad \forall n$ — (ii)

$\therefore$ (i) & (ii) $\Rightarrow$

$k \le a_n \le K \quad \forall n.$

$\therefore \langle a_n \rangle$ is bdd.

Converse is not true

choose $\varepsilon > \frac{1}{3}|l-l'|$

$\therefore \langle a_n \rangle \to l$ &

$\langle a_n \rangle \to l'$

$\therefore$ corr. to $\varepsilon = \frac{1}{3}|l-l'| > 0$

$\exists$ +ve integers m_1 & m_2 s.t.

$$|a_n - l| < \varepsilon \quad \forall n \geq m_1 \quad \text{—(i)}$$

$$|a_n - l'| < \varepsilon \quad \forall n \geq m_2 \quad \text{—(ii)}$$

Th: A sequence can not converge to more than one limit

Proof: Let, if possible, a convergent seq. $\langle a_n \rangle$ converge to the two limits l and l'.

$l \neq l'$

$\therefore |l-l'| > 0$

$\therefore |l - l'| < 2\varepsilon$

$$|l - l'| < 2 \times \frac{1}{3}|l - l'|$$

i.e $|l - l'| < \frac{2}{3}|l - l'|$

This is a contradiction.

$\therefore$ our supposition is wrong.

$\underline{l = l'}$

Let $m = \max\{m_1, m_2\}$.

$\therefore$ (I) and (II) hold for all $n \geq m$.

$$|l - l'| = |(l - a_n) + (a_n - l')|$$

$$\leq |l - a_n| + |a_n - l'|$$

$$= |a_n - l| + |a_n - l'|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

$$\Rightarrow a = (1+h)^n$$

$$= 1 + nh + \frac{n(n-1)}{\lfloor 2} h^2 + \cdots + h^n$$

$$> 1 + nh \quad (\because h > 0)$$

i.e. $a > 1 + nh$

$$\Rightarrow a - 1 > nh$$

$$\Rightarrow h < \frac{a-1}{n} \quad \text{--- (1)}$$

Th: Every convergent sequence is bounded and has a unique limit. (Prove the two previous theorems).

Proof: By defn. show that

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad a > 0.$$

Soln: Case $a > 1$

Let $a^{\frac{1}{n}} = 1 + h$, $h > 0$

$\therefore \lim a^{\frac{1}{n}} = 1$ for $a > 1$

$\boxed{a = 1}$ $\lim a^{\frac{1}{n}} = 1^{\frac{1}{n}} = 1$.

obvious.

$\boxed{0 < a < 1}$ Let $b = \frac{1}{a}$.

$\because 0 < a < 1$

$\therefore b > 1$

As in the case of $a > 1$,

Let $\varepsilon > 0$ be any number

$|h| = h < \frac{a-1}{n} < \varepsilon$ if $n > \frac{a-1}{\varepsilon}$.

using (1)

Let m be a +ve int greater than $\frac{a-1}{\varepsilon}$.

$\therefore |h| < \varepsilon, \ \forall n \geq m$

i.e. $|a^{\frac{1}{n}} - 1| < \varepsilon, \ \forall n \geq m$

Prob: prove $\lim \sqrt[n]{n} = 1$

$\rightarrow$ let $n^{\frac{1}{n}} = 1+h$ $\quad h \geq 0$

$\Rightarrow n = (1+h)^n$

$= 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots + h^n$

$\Rightarrow n > \frac{n(n-1)}{2} h^2$ $\quad \forall n \geq 2$

$\Rightarrow h^2 < \frac{2}{n-1}$

We can prove.

$\lim b^{1/n} = 1$

i.e $\lim \left(\frac{1}{a}\right)^{1/n} = 1$

i.e. $\lim \frac{1}{a^{1/n}} = 1$

i.e. $\lim a^{\frac{1}{n}} = 1$

Let m be a +ve integer greater than $1+\frac{2}{\varepsilon^2}$

$\therefore \; |h| < \varepsilon \quad \forall n \geq m$

i.e. $\left|n^{\frac{1}{n}}-1\right| < \varepsilon \quad \forall n \geq m$

$\therefore$ for $\varepsilon > 0$,

$\left|n^{\frac{1}{n}}-1\right| < \varepsilon \quad \forall n \geq m$

$\boxed{\therefore \lim n^{\frac{1}{n}} = 1}$

$$h^2 < \frac{2}{n-1}$$

$$\Rightarrow |h| < \sqrt{\frac{2}{n-1}}$$

Let $\varepsilon > 0$ be any number

$|h| < \sqrt{\frac{2}{n-1}} < \varepsilon$

$x^2 < a^2 \iff |x| < a$

if $\frac{n-1}{2} > \frac{1}{\varepsilon^2}$

i.e. if $n > 1 + \frac{2}{\varepsilon^2}$

(iv) $\lim\left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n} = \frac{a}{b}$.

$b_n \neq 0$

$b \neq 0$

Remark: Converse is not true.

Consider the following example.

(1) $a_n = n^2$, $b_n = -n^2$

$\langle a_n + b_n \rangle \to 0$

Th: If $\langle a_n \rangle$, $\langle b_n \rangle$ be two sequences such that $\lim a_n = a$ & $\lim b_n = b$ then

(i) $\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$
$= a \pm b$.

(ii) $\lim(a_n \cdot b_n) = (\lim a_n) \cdot \lim(b_n)$
$= a \cdot b$.

(iii) $\lim\left(\frac{1}{a_n}\right) = \frac{1}{\lim a_n} = \frac{1}{a}$, $a_n \neq 0$, $a \neq 0$

But $\langle a_n \rangle$ and $\langle b_n \rangle$ are oscillatory seq.

<u>prob</u>: prove that

$$\lim_{n\to\infty} \frac{2n^2+3}{3n^2+5n} = \frac{2}{3}$$

<u>Sol^n</u>

$$\lim_{n\to\infty} \frac{2+\frac{3}{n^2}}{3+\frac{5}{n}}$$

$$\left\langle \frac{a_n}{b_n} \right\rangle \to -1$$

But $\langle a_n \rangle$ and $\langle b_n \rangle$ are divergent.

11 $a_n = b_n = (-1)^n$.

$$\langle a_n - b_n \rangle \longrightarrow 0$$

$$\langle a_n \cdot b_n \rangle \longrightarrow 1$$

$$\left\langle \frac{a_n}{b_n} \right\rangle \longrightarrow 1$$

Prob: If $\lim a_n = a$ and $a \neq 0$, show that there exists a +ve int. m and a positive number K s.t.

$$|a_n| > K \quad \forall n \geq m.$$

Solⁿ:

$a \neq 0$

$\therefore |a| > 0$

Let $\varepsilon = \frac{1}{2}|a| > 0$

$$= \frac{\lim_{n\to\infty}\left(2 + \frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(3 + \frac{5}{n}\right)}$$

$$= \frac{2}{3}$$

$|a| - \frac{1}{2}|a| < a_n \quad , \forall n \geq m$

$a_n > \frac{1}{2}|a| , \forall n \geq m$

i.e. $a_n > K , \quad \forall n \geq m$

$(K = \frac{1}{2}|a|)$

$\because \langle a_n \rangle \to a$

$\therefore$ corr. to the $\varepsilon = \frac{1}{2}|a| > 0$

$\exists$ a +ve int m s.t.

$|a_n - a| < \varepsilon \quad \forall n \geq m.$

$|a| - |a_n| \leq |a_n - a| < \varepsilon$

$\forall n \geq m$

$|a| - |a_n| < \varepsilon , \quad \forall n \geq m$

$|a| - |a_n| < \frac{1}{2}|a| , \quad \forall n \geq m$

$||a_n| - |a|| \le |a_n - a| < \varepsilon$

$\forall n \geq m$

using ①

$\therefore$

$||a_n| - |a|| < \varepsilon$

$\forall n \geq m$

$\therefore \lim |a_n| = |a|$

Prob: prove that

$a_n \to a \Rightarrow |a_n| \to |a|$.

Converse is not true.

Solⁿ: Let $\varepsilon > 0$ be given.

$\because a_n \to a$

$\therefore$ cor. to $\varepsilon > 0$, $\exists$ a +ve int. m s.t.

$|a_n - a| < \varepsilon \quad \forall n \geq m$ — ①

Prob: Show that if $a_n \to a$ then $a_n^2 \to a^2$. Is the converse true?

$\to$ $\because$ $\langle a_n \rangle$ is cgt.

$\therefore$ seq is bdd.

$\therefore$ $\exists$ a +ve real no. K

s.t. $|a_n| \leq K \quad \forall n$. —(1)

Consider

$$|a_n^2 - a^2| = |a_n - a| \cdot |a_n + a|$$

Converse is not true.

Consider the example

$$\langle (-1)^n \rangle$$

Let $a_n = (-1)^n$

$$\langle |a_n| \rangle \to 1$$

But $\langle a_n \rangle$ is not convergent.

$|a_n - a| < \frac{\varepsilon}{K+|a|}, \forall n \geq m.$

using this in (11)

$$|a_n^2 - a^2| < \varepsilon$$

$\forall n \geq m$

$$\therefore \lim a_n^2 = a^2$$

Converse is not true.

Consider the example.

$$|a_n^2 - a^2| \leq (|a_n| + |a|) \cdot |a_n - a|$$

$$\text{i.e. } |a_n^2 - a^2| \leq (K + |a|)|a_n - a|$$

(11)

Let $\varepsilon > 0$ be given. using (1)

$\because a_n \to a$

$\therefore$ corr. to $\frac{\varepsilon}{K+|a|} > 0$, $\exists$ a +ve int. m s.t

Show that the sequence $\langle s_n \rangle$ and $\langle t_n \rangle$ are cgt. and that

$\lim s_n = \max\{a, b\}$ and

$\lim t_n = \min\{a, b\}$.

proof: we know that

$$\max(x, y) = \frac{1}{2}(x+y) + \frac{1}{2}|x-y| \quad (1)$$

$$\min(x, y) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| \quad (11)$$

$\langle a_n \rangle \qquad a_n = (-1)^n$

$\therefore \langle a_n^2 \rangle \to 1$

But $\langle a_n \rangle$ is not cgt.

proof: Given that $\lim a_n = a$ and $\lim b_n = b$, $\langle s_n \rangle$ and $\langle t_n \rangle$ are two sequence, where $s_n = \max\{a_n, b_n\}$ and $t_n = \min\{a_n, b_n\}$.

$\therefore \lim s_n = \max(a, b)$

$\therefore \langle s_n \rangle$ is cgt.

and $\lim s_n = \max\{a, b\}$.

(11) Similar.
(HW)

Put $x = a_n$, $y = b_n$ in (1)

$$\underline{\max\{a_n, b_n\}} = \frac{1}{2}(a_n + b_n) + \frac{1}{2}|a_n - b_n|$$

$$s_n = \frac{1}{2}(a_n + b_n) + \frac{1}{2}|a_n - b_n|$$

$$\lim s_n = \frac{1}{2}\lim(a_n + b_n) + \frac{1}{2}\lim|a_n - b_n|$$

$$= \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$$