KGS IAS

Maths Optional

By Dhrub Singh Sir

Step 3: The given eqn. reduces to

$$\frac{d^2y}{dz^2} + q_1 y = R_1$$

To solve $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = R$

by changing the independent variable.

Working rule

Step 1: Compute $z = \int e^{-\int p\,dx}\,dx$

Step 2: $\frac{dz}{dx}$

$$q_1 = \frac{q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let $Z = \int \left[e^{\int -P\,dx} \right] dx$

$= \int \left[e^{-2\int \frac{1}{x} dx} \right] dx$

$= \int e^{-2\log x} \, dx$

$= \int e^{\log \frac{1}{x^2}} \, dx$

$= \int \frac{1}{x^2} \, dx$

$= -\frac{1}{x}.$

Prob: Solve

$$x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + n^2 y = 0$$

$$\rightarrow \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{n^2}{x^4} y = 0 \quad (1)$$

Comparing with ...

$$P = \frac{2}{x}, \quad Q = \frac{n^2}{x^4}, \quad R = 0$$

Now ① reduces to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

i.e. $$\frac{d^2y}{dz^2} + n^2 y = 0$$

A.E. : $m^2 + n^2 = 0$

$\Rightarrow m^2 = -n^2$

$\Rightarrow m = \pm n i$

$\therefore y = c_1 \cos nz + c_2 \sin nz$

$= c_1 \cos\left(-\frac{n}{x}\right) + c_2 \sin\left(-\frac{n}{x}\right)$

$$\frac{dz}{dx} = \frac{d\left(-\frac{1}{x}\right)}{dx}$$

$$= \frac{1}{x^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{n^2}{\cancel{x^4}}}{\frac{1}{\cancel{x^4}}}$$

$$= n^2$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

Prob: Solve

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}+4x^3y=x^5$$

$$\rightarrow \frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}+4x^2y=x^4 \quad (1)$$

Comparing it with,

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

$$P=-\frac{1}{x},\quad Q=4x^2,\quad R=x^4.$$

Let $$Z=\int\left[e^{-\int P\,dx}\right]dx$$

$$=\int\left[e^{\int\frac{1}{x}dx}\right]dx$$

$$=\int e^{\log x}\,dx$$

$$=\int x\,dx=\frac{x^2}{2}.$$

$$\frac{dZ}{dx}=x$$

(1) reduces to.

$$\frac{d^2y}{dz^2}+4y=2z$$

$\underline{A.E}$: $m^2+4=0$

$m^2=-4$

$\Rightarrow m=\pm 2i$

$C.F.=c_1\cos 2z+c_2\sin 2z.$

$=c_1\cos(x^2)+c_2\sin(x^2)$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$=\frac{4x^2}{x^2}=4$$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$=\frac{x^4}{x^2}=x^2=2z.$$

$\therefore y = CF + PI$

$= c_1 \cos(x^2) + c_2 \sin(x^2)$

$+ \frac{x^2}{4}.$

$P.I. = \frac{1}{D^2+4}(2z)$

$= \frac{2}{4} \cdot \frac{1}{\left(1+\frac{D^2}{4}\right)} z$

$= \frac{1}{2} \cdot \left[1+\frac{D^2}{4}\right]^{-1} z$

$= \frac{1}{2}[1] z$

$= \frac{1}{2} z = \frac{x^2}{4}.$

Let $z = \int \left[e^{-\int p\,dx} \right] dx$

$= \int \left[e^{\int -\frac{\sin x}{\cos x}\,dx} \right] dx$

$= \int e^{\log \cos x}\, dx$

$= \int \cos x\, dx = \sin x$

$\therefore \dfrac{dz}{dx} = \cos x$

<u>Prob:</u> Solve

$$\cos x \cdot \frac{d^2y}{dx^2} + \sin x \cdot \frac{dy}{dx} - 2(\cos^3 x)y = 2\cos^5 x$$

$$\longrightarrow \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2\cos^2 x\, y = 2\cos^4 x \quad \text{--- (1)}$$

Comparing,

$P = \tan x$, $Q = -2\cos^2 x$

$R = 2\cos^4 x$

Eqn. (1) reduces to

$$\frac{d^2y}{dz^2} - 2y = 2(1-z^2)$$

A.E. $m^2 - 2 = 0$

$\Rightarrow m = \pm\sqrt{2}$

$$C.F. = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$$

$$= c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{-2\cancel{\cos^2 x}}{\cancel{\cos^2 x}} = -2$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x$$

$$= 2(1-\sin^2 x)$$

$$= 2(1-z^2)$$

$= (\sin x)^2 = \sin^2 x$

$\therefore y = CF + PI$

$= c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x.$

HW

Prob: Solve

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin(x^2)$$

Ans: $y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin(x^2)$

$$P.I. = \frac{1}{D^2-2}\, 2(1-z^2)$$

$$= \frac{2}{-2} \cdot \frac{1}{\left(1-\frac{D^2}{2}\right)}(1-z^2)$$

$$= -1\left[1-\frac{D^2}{2}\right]^{-1}(1-z^2)$$

$$= -1\left[1+\frac{D^2}{2}\right](1-z^2)$$

$$= -1\left[\cancel{1}-z^2-\cancel{1}\right]$$

$$= z^2$$

Solⁿ. of

$$f(D)y = 0$$

$$W(y_1, y_2, \ldots y_n)(x)$$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Variation of Parameters

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = R \longrightarrow (1)$$

Linear diff. eqn. of order 'n'.

Wronskian: If $y = y_1(x)$, $y = y_2(x)$, $\cdots$ $y = y_n(x)$

possesses n L.I.
sol^n (distinct).

Working rule:

Step I: write the given diff eq^n in Standard form,

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

Note (i) If $W(x)\neq 0$ then the sol^n $y_1, y_2 \ldots y_n$ are L.I.

(ii) If $W(x)=0$ then the sol^n $y_1, y_2, \ldots y_n$ are L.D.

(iii) The nth order L.D.E
$f(D)\,y=0$

Step 4:

$$W(x) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix}$$

Step 5:

$$A = \int \frac{-VR}{W}\,dx$$

$$B = \int \frac{uR}{w}\,dx$$

Gen soln. $y = y_c + y_p$

Step 2 Find the soln. of

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

Let it be

$$y_c = c_1\, u(x) + c_2\, V(x)$$

Step 3: Let P.I. of the original D. Eqn. is

$$y_p = A \cdot u(x) + B \cdot V(x)$$

where A and B are fn. of x.

$y_c = c_1 \underbrace{\cos ax}_{u} + c_2 \underbrace{\sin ax}_{v}$.

$u = \cos ax \qquad v = \sin ax$

Let $y_p = A\cos ax + B\sin ax$

be P.I. of (1)

A and B are fn. of x.

$$W(x) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$= a$

Prob: Solve $(D^2+a^2)y = \tan ax$

by the method of variation

of parameters.

Sol^n: $D^2y + a^2y = \tan ax$ — (1)

$P = 0,\ Q = a^2,\ R = \tan ax$

Associated homogeneous eqn.

$(D^2+a^2)y = 0$

A.E. $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

$$= \frac{-1}{a}\left[\int \sec ax\,dx - \int \cos ax\,dx\right.$$

$$= \frac{-1}{a}\left[\frac{1}{a}\cdot\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$$= \frac{-1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]$$

$$B = \int \frac{uR}{W}\,dx = \int \frac{\cos ax \cdot \tan ax}{a}\,dx$$

$$= \frac{1}{a}\int \sin ax\,dx$$

$$= -\frac{\cos ax}{a^2}$$

$$A = \int \frac{-vR}{W}\,dx$$

$$= \int \frac{-\sin ax \cdot \tan ax}{a}\,dx$$

$$= \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$= \frac{-1}{a}\int \frac{(1-\cos^2 ax)}{\cos ax}\,dx$$

Prob: Solve

$$\left[(x-1)D^2 - xD + 1\right]y = (x-1)^2$$

by the method of variation of parameters.

$\longrightarrow$

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

$$y_p = -\frac{1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]\cdot \cos ax$$

$$-\frac{1}{a^2}\cos ax \cdot \sin ax$$

$\therefore$ gen solⁿ is

$$y = y_c + y_p$$

Prob: solve $y'' + 4y = 4\tan 2x$.

HW

$$P+Q=\frac{-x}{x-1}+\frac{1}{x-1}$$

$$=\frac{-x+1}{x-1}$$

$$=-1$$

$\therefore P+Q+1=0$

$\therefore y=e^x$ is the part of C.F. of (ii)

$$\boxed{u=e^x}$$

$$\frac{d^2y}{dx^2}-\frac{x}{x-1}\frac{dy}{dx}+\frac{y}{x-1}=(x-1) \quad \text{—— (i)}$$

Comparing,

$$P=\frac{-x}{x-1},\quad Q=\frac{1}{x-1}$$

$$R=x-1$$

Associated homogeneous eqn. is

$$\frac{d^2y}{dx^2}-\frac{x}{x-1}\frac{dy}{dx}+\frac{y}{x-1}=0 \quad \text{—— (ii)}$$

$$\frac{d^2u}{dx^2}+\left[\frac{-x+2x-2}{x-1}\right]\frac{du}{dx}=0$$

$$\Rightarrow \frac{d^2u}{dx^2}+\left[\frac{x-1-1}{x-1}\right]\frac{du}{dx}=0$$

$$\Rightarrow \frac{dz}{dx}+\left[1-\frac{1}{x-1}\right]z=0$$

$$\Rightarrow \frac{dz}{z}+\left[1-\frac{1}{x-1}\right]dx=0 \qquad z=\frac{du}{dx}$$

$\therefore y=e^x\cdot v$ is the gen. soln. of (11), where v is given by

$$\frac{d^2u}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{du}{dx}=0$$

$$\Rightarrow \frac{d^2u}{dx^2}+\left[-\frac{x}{x-1}+\frac{2}{e^x}\times e^x\right]\frac{du}{dx}=0$$

$$\Rightarrow \frac{d^2u}{dx^2}+\left[-\frac{x}{x-1}+2\right]\frac{du}{dx}=0$$

$$du = c_1 e^{-x} \cdot x\, dx - c_1 e^{-x}\, dx$$

Int.

$$v = c_1\left[-x e^{-x} + \int e^{-x}\, dx\right] + c_1 e^{-x}$$

$$= c_1\left[-x e^{-x} - \cancel{e^{-x}}\right] + c_2 \cancel{+ c_1 e^{-x}}$$

$$+ c_2$$

$$= -x e^{-x} \cdot c_1 + c_2$$

$$\therefore y_c = e^{x}\left(-x e^{-x} c_1 + c_2\right)$$

Int.

$$\log z + x - \log(x-1) = \text{const}$$

$$\Rightarrow \log \frac{z}{x-1} = -x + \text{const}$$

$$\Rightarrow \frac{z}{x-1} = e^{-x} \cdot c_1$$

$$\Rightarrow z = e^{-x}(x-1) \cdot c_1$$

$$\Rightarrow \frac{du}{dx} = e^{-x}(x-1) c_1$$

$$A = \int \frac{-VR}{W}\,dx$$

$$= \int \frac{-e^{x}\cdot(x-1)}{e^{x}(1-x)}\,dx$$

$$= \int dx = x$$

$$B = \int \frac{UR}{W}\,dx = \int \frac{-x(x-1)}{e^{x}(1-x)}\,dx.$$

$$= \int x e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

$$= -e^{-x}(x+1)$$

$$y_c = \underbrace{-x}_{U}\, c_1 + \underbrace{e^{x}}_{V} c_2$$

$$U = -x \qquad V = e^{x}$$

$$y_p = A\cdot(-x) + B\,e^{x} \quad \text{is}$$

the P.I. of 1

$$W = \begin{vmatrix} -x & e^{x} \\ -1 & e^{x} \end{vmatrix} = -xe^{x} + e^{x}$$

$$= e^{x}(1-x)$$