KGS IAS

Maths Optional

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Clairaut's form

gen soln. is

$$y = cx \pm a\sqrt{c^2-1} \quad \text{--- (1)}$$

$$\Rightarrow y - cx = \pm a\sqrt{c^2-1}$$

Squaring,

$$(y-cx)^2 = a^2(c^2-1)$$

$$\Rightarrow c^2x^2 + y^2 - 2cxy = a^2c^2 - a^2$$

2022 HW Prob:

$$(x^2-a^2)p^2 - 2xyp + y^2 + a^2 = 0$$

$$\Rightarrow x^2p^2 - 2xyp + y^2 = a^2p^2 - a^2$$

$$\Rightarrow (px-y)^2 = a^2(p^2-1)$$

$$\Rightarrow (y-px)^2 = a^2(p^2-1)$$

$$\Rightarrow y - px = \pm a\sqrt{p^2-1}$$

$$\Rightarrow y = px \pm a\sqrt{p^2-1} \quad \text{--- (1)}$$

$$a^2(x^2-y^2)-a^4=0$$

$$\Rightarrow x^2-y^2-a^2=0$$

$$\Rightarrow \boxed{x^2-y^2=a^2}$$

common factor b/w c-disc. & p-disc.

$$y^2=x^2-a^2$$

It satisfies D.E. (Check it!)

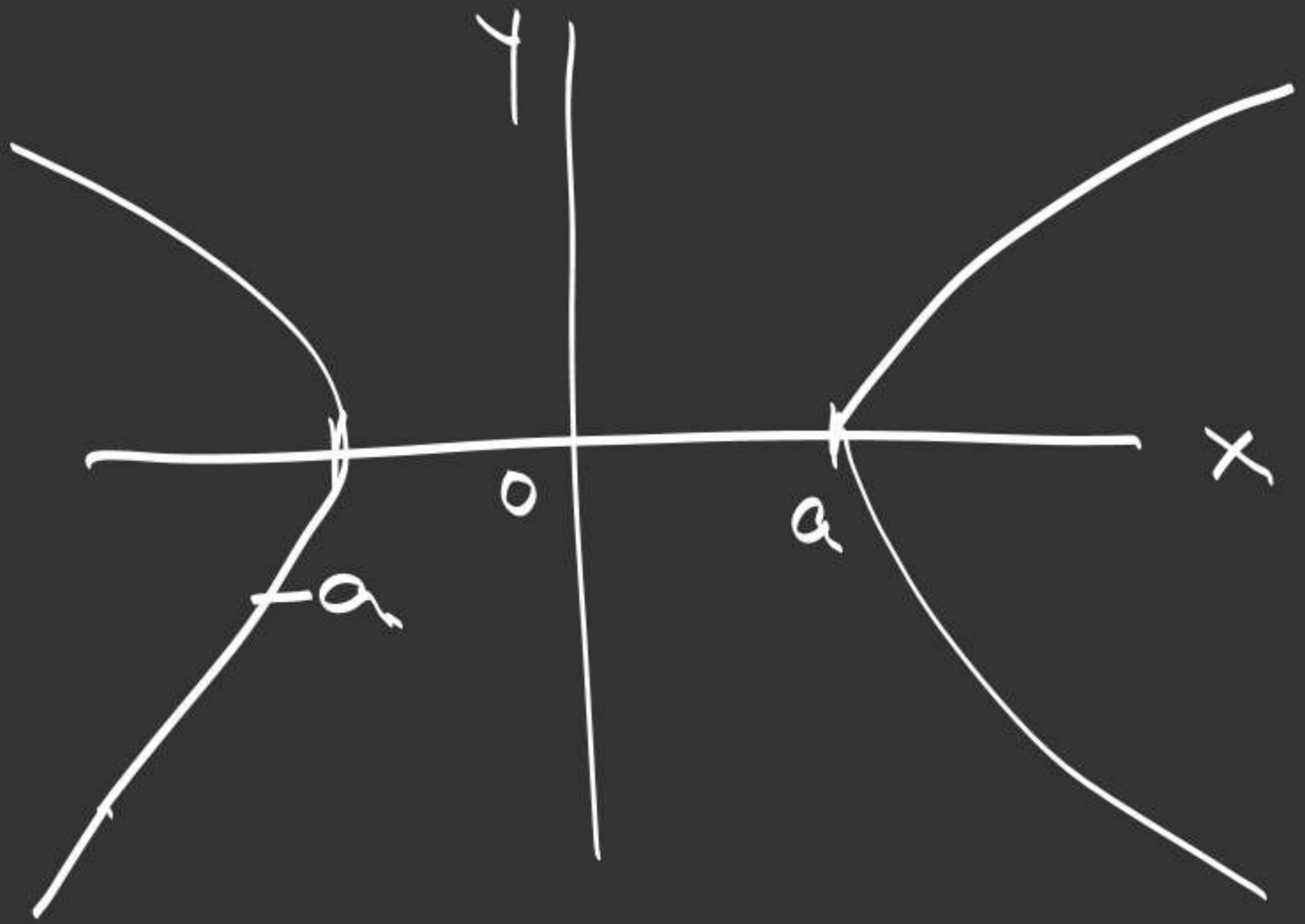
$$\Rightarrow c^2(x^2-a^2)-2cxy+y^2+a^2=0$$

<u>C-Disc</u> (p-Disc).

$$(-2xy)^2-4(x^2-a^2)(y^2+a^2)=0$$

$$\Rightarrow 4\cancel{x^2y^2}-4\left[\cancel{x^2y^2}+x^2a^2-a^2y^2-a^4\right]=0$$

$$\Rightarrow -\left[a^2(x^2-y^2)-a^4\right]=0$$



$\therefore x^2 - y^2 = a^2$ — S.S.

$y = cx \pm a\sqrt{c^2-1}$

Rectangular Hyperbola

Envelope to the

Family of st. lines

$$\frac{dy}{dx} = \pm\frac{2}{3}\frac{\sqrt{3-y}}{(2-y)}$$

$$\pm\frac{3}{2}\times\frac{2-y}{\sqrt{3-y}}\,dy = dx$$

int

$$\pm\frac{3}{2}\left[\int\frac{3-y-1}{\sqrt{3-y}}\,dy\right] = x+c$$

2020 prob: Find the general and singular sol^n. of the D.E.

$$9p^2(2-y)^2 = 4(3-y) \quad \text{--- } 1$$

$$\Rightarrow \quad p^2 = \frac{4}{9}\frac{(3-y)}{(2-y)^2}$$

$$\Rightarrow \quad p = \pm\frac{2}{3}\frac{\sqrt{3-y}}{(2-y)}$$

$$\Rightarrow \pm\sqrt{3-y}\;\; y = x+c$$

squaring.

$$(x+c)^2 = y^2(3-y)$$

$$\Rightarrow c^2 + 2cx + x^2 - y^2(3-y) = 0$$

C- Disc:

$$\cancel{(2x)^2} - 4\left[\cancel{x^2} - y^2(3-y)\right] = 0$$

$$4y^2(3-y) = 0$$

$$\pm\frac{3}{2}\left[\int\sqrt{3-y}\,dy - \int\frac{dy}{\sqrt{3-y}}\right] = x+c$$

$$\Rightarrow \pm\frac{3}{2}\left[-\frac{2}{3}(3-y)^{\frac{3}{2}} + 2\sqrt{3-y}\right] = x+c$$

$$\Rightarrow \pm\frac{3}{\cancel{2}}\times\cancel{2}\sqrt{3-y}\left[-\frac{1}{3}(3-y)+1\right] = x+c$$

$$\Rightarrow \pm\cancel{3}\sqrt{3-y}\left[\frac{-3+y+3}{\cancel{3}}\right] = x+c$$

$(y-2)^2(y-3)=0$ —— (iii)

From (ii) & (iii)

$y-3=0$ is the common factor

$\Rightarrow y=3$

$\Rightarrow \dfrac{dy}{dx}=0 \Rightarrow p=0$

$\therefore y=3$ satisfies the D.E. (Check it!!)

$y^2(y-3)=0$ —— (ii)

(i) $\Rightarrow$

$9p^2(2-y)^2-4(3-y)=0$

p-disc:

$(0)^2+4\times 9(2-y)^2\times 4(3-y)=0$

$\Rightarrow (2-y)^2(3-y)=0$

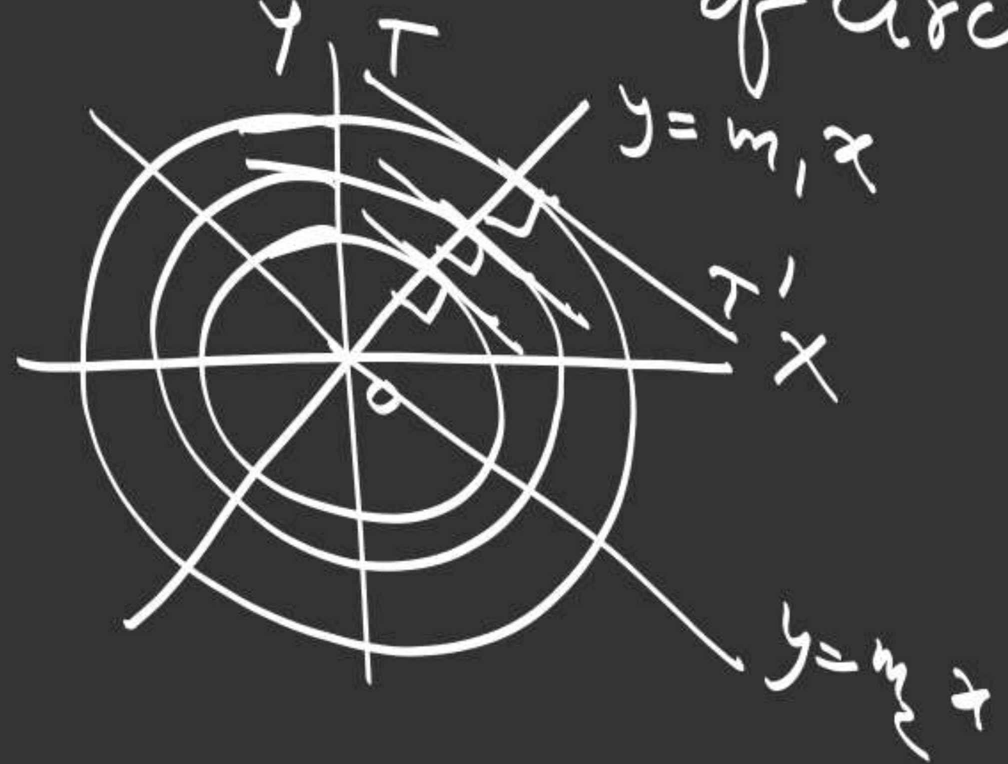
$y = \underline{mx}$ (Family of st. lines)

$\Rightarrow$ orthogonal trajectories to the family of circles represented by $x^2+y^2=a^2$.

$\therefore y=3$ — $\underline{\text{S.S.}}$

orthogonal trajectories

<u>Ex:</u> $x^2+y^2=a^2$ — Family of circles.



Working rule to find orthogonal traj.

(1) $f(x, y, c) = 0$ — (i)

$\downarrow$

Form the D.E. of this family.

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{— (ii)}$$

(2) Replace $\frac{dy}{dx}$ in (ii) with $-\frac{1}{\frac{dy}{dx}}$ (i.e. $-\frac{dx}{dy}$)

$$F\left(x, y, -\frac{dx}{dy}\right) = 0$$

Solve

$g(x, y, c_1) = 0$ — orth traj.

$\Rightarrow y\dfrac{dy}{dx}+x=0$ —— (11)

D.E. of (1)

orthogonal traj to (1)

Replace $\dfrac{dy}{dx}$ in (11) with

$-\dfrac{1}{\frac{dy}{dx}}$ (i.e. $-\dfrac{dx}{dy}$)

$y\left(-\dfrac{dx}{dy}\right)+x=0$

prob: Find the orthogonal trajectories of the family of curves $x^2+y^2=a^2$, a is a parameter.

$\longrightarrow \quad x^2+y^2=a^2$ —— (1)

Diff. w.r.t. x

$2x+2y\dfrac{dy}{dx}=0$

prob: Find the family of orth. traj. of $ay^2 = x^3$, a is a parameter.

$\rightarrow$ $ay^2 = x^3$ —— (1)

Diff. w.r.t. 'x',

$a \cdot 2y\, y' = 3x^2$

$a = \frac{3}{2} \frac{x^2}{y y'}$

put in (1)

$\Rightarrow x\, dy - y\, dx = 0$

$\Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$

Int.

$\log y - \log x = \text{const}$

$\Rightarrow \log \frac{y}{x} = \text{const.}$

$\Rightarrow \frac{y}{x} = c$

$\Rightarrow y = cx$

$$3y = 2x\left(-\frac{dx}{dy}\right)$$

$$\Rightarrow 3y\,dy = -2x\,dx$$

Int.

$$\frac{3y^2}{2} = -x^2 + \text{const}$$

$$\Rightarrow 3y^2 + 2x^2 = \text{const}$$

$$\Rightarrow \boxed{\frac{x^2}{3} + \frac{y^2}{2} = C}$$

$$\frac{3}{2}\,\frac{x^2}{yy'}\cdot y^2 = x^3$$

$$\Rightarrow 3x^2y^2 = 2x^3 y y'$$

$$\Rightarrow 3y = 2x\,y'$$

$$\Rightarrow 3y = 2x\frac{dy}{dx} \quad \text{--- (1)}$$

For orth. traj,

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in (1)

Diff. w.r.t. 'x',

$$2y\frac{dy}{dx} = 4a$$

Using this in (1)

$$y^2 = 2y\frac{dy}{dx} + \left(y\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2 = 2y\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \quad \text{--- (ii)}$$

D.E. of (1)

Self-orthogonal family of curves.

prob: Show that the system of confocal and co-axial parabolas $y^2 = 4a(x+a)$ is self orthogonal.

$$\longrightarrow \quad y^2 = 4ax + 4a^2 \quad \text{--- (1)}$$

$$y^2\left(\frac{dy}{dx}\right)^2 = -2y\frac{dy}{dx} + y^2$$

$$\Rightarrow y^2 = 2y\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \quad \text{(iii)}$$

(ii) & (iii) are same.

$\therefore$ Family of curves given by (1) is self orthogonal.

For orth. traj.,

Replace $\frac{dy}{dx}$ in (ii) with $-\frac{1}{\frac{dy}{dx}}$

$$y^2 = 2y\cdot\frac{-1}{\frac{dy}{dx}} + y^2\left(\frac{-1}{\frac{dy}{dx}}\right)^2$$

$$y^2 = -\frac{2y}{\frac{dy}{dx}} + \frac{y^2}{\left(\frac{dy}{dx}\right)^2}$$

Diff. ① w.r.t. 'x'

$$\frac{\cancel{2}x}{a^2+\lambda}+\frac{\cancel{2}y\cdot y'}{b^2+\lambda}=0$$

$$\Rightarrow \lambda = \frac{-(b^2x+a^2yy')}{x+yy'}$$

$$a^2+\lambda = a^2-\frac{b^2x+a^2yy'}{x+yy'}$$

$$=\frac{(a^2-b^2)x}{x+yy'}$$

prob: Show that the system of confocal conics (ellipse)

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$ is

self orthogonal.

$$\rightarrow \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \longrightarrow ①$$

$$\left(x-\frac{y}{y'}\right)(x+yy')=a^2-b^2 \quad (iii)$$

(ii) and (iii) are same.

$\therefore$ (i) is self-orth.

$$b^2+\lambda=\frac{-(a^2-b^2)yy'}{x+yy'}$$

Now (i) $\Rightarrow$

$$(x+yy')\left(x-\frac{y}{y'}\right)=a^2-b^2 \quad (ii)$$

D.E. of (i)

For orth. traj.

Replace y' in (ii) with $-\frac{1}{y'}$.

② For orth. traj:

Replace $\frac{dr}{d\theta}$ in (II)

with $\boxed{-r^2 \frac{d\theta}{dr}}$

$$F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$$

Solve

Orthogonal traj in polar Co-ordinates:

① $f(r, \theta, c) = 0$ —— (I)

For D.E.

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{—— (II)}$$

2013 prob: Find the orth traj of the family of curves $r^n = a\sin n\theta$, a is the parameter.

$\rightarrow$ $r^n = a\sin n\theta$ — ①

Diff. w.r.t. 'θ'

$n r^{n-1}\dfrac{dr}{d\theta} = an\cos n\theta$

2011 prob: Find the orth. traj of family of the curves $r = a(1-\cos\theta)$, a is a parameter.

$\longrightarrow$ $r = a - a\cos\theta$ — ①

Ans: $r = b(1+\cos\theta)$

$$\frac{1}{r}\,dr = -\tan n\theta\, d\theta$$

$$\Rightarrow \frac{1}{r}\,dr + \frac{\sin n\theta}{\cos n\theta}\, d\theta = 0$$

Int

$$\log r - \frac{1}{n}\log\cos n\theta = \text{const}$$

$$\Rightarrow n\log r - \log\cos n\theta = \text{const}$$

$$\Rightarrow \log\frac{r^n}{\cos n\theta} = \text{const}$$

(1) $\Rightarrow$

$$r = \tan n\theta \cdot \frac{dr}{d\theta} \quad \text{— (1)}$$

For orth. traj.

Replace $\frac{dr}{d\theta}$ in (1)

with $-r^2\frac{d\theta}{dr}$

$$r = \tan n\theta\left(-r^2\frac{d\theta}{dr}\right)$$

$$\frac{r^n}{\cos n\theta} = b$$

$$\Rightarrow \boxed{r^n = b\cos n\theta}$$