KGS IAS

Maths Optional

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Calculus & Real Analysis

- Double points
- Asymptotes
- curve tracing

→ calculus of several variables

- Limit & conti.
- Differentiability.

I

- Mean value theorem.
- Monotone functions.
- Maxima & Minima
- Indeterminate forms.

II

- Real numbers, open set, closed set, Limit point.
- Real sequence.
- Infinite series.
- Limit & continuity
- Differentiability.

- Double & Triple integral (Area, volume, surface —)

- Maxima & Minima.
- Lagrange's method of undetermined multipliers
- Riemann Integration.
- uniform convergence
- β & Γ fn.
- Improper integral.

Rational Number

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \; q \neq 0 \right\}$$

Irrational number

$$\mathbb{Q}^* = \mathbb{R} \sim \mathbb{Q}$$

Cannot be put in the form of $\frac{p}{q}$, $p, q \in \mathbb{Z}$.

Real Numbers

Natural number

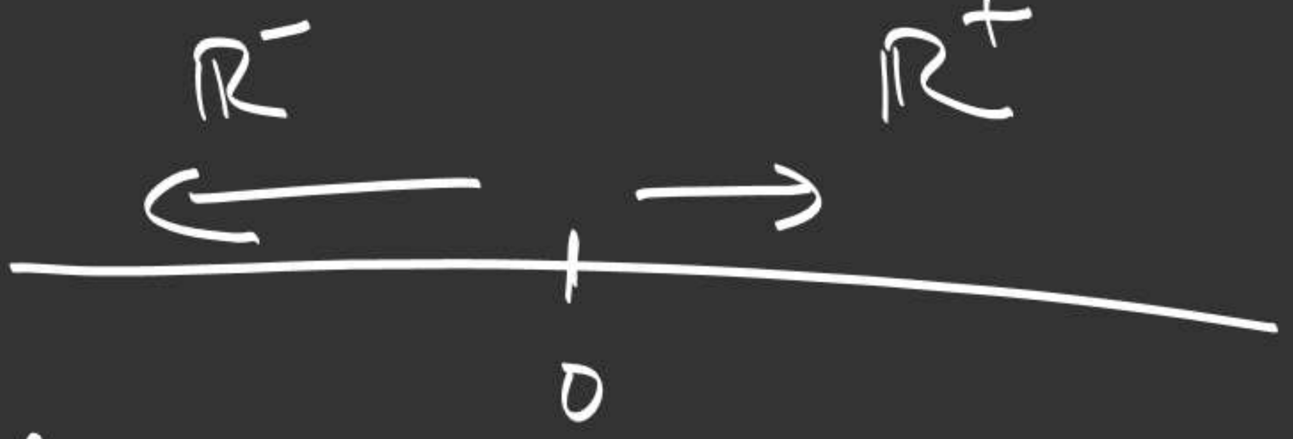
$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Integers

$$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$$

Whole No: $W = \{0, 1, 2, \ldots\}$

Real line



Rational No.:

$$\underbrace{\frac{4}{5} = 0.8}_{\text{Terminating Decimal}} \qquad \underbrace{\frac{2}{3} = 0.6666\ldots}_{\text{Non-terminating recurring dec.}}$$

Ex: $\sqrt{2}, \sqrt{3}, 2+\sqrt{5}, 2-\sqrt{5}, -\sqrt{3}, e, \pi, \ldots$

Reals: Rationals and Irrationals together make reals.

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^*$$

$\forall \to$ for all

$\not\forall \to$ For some

$\exists \to$ there exists (At least one)

$\Longrightarrow \to$ implies

$\Longleftrightarrow \to$ implies and implied by

iff

Irrational number

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Non-terminating non-recurring dec.

Note ① $\mathbb{N} \subset W \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\mathbb{Q}^* \subseteq \mathbb{R}$.

(II) order axioms.

(III) Completeness axioms.

Field Axioms:

(A) $(\mathbb{R}, +, \cdot)$

(I) $a+b \in \mathbb{R} \quad \forall a, b \in \mathbb{R}$

(closure prop)

(II) $a+b = b+a \quad \forall a, b \in \mathbb{R}$

ϵ or ε — epsilon

δ — Delta

$\epsilon > 0 \longrightarrow$ be a very small the Real No.

Some important properties of real numbers in the form of axioms.

(I) Field Axioms.

(V) Existence of addition inverse.

$$a+(-a)=(-a)+a=0$$

$\underbrace{-a}_{\text{inverse}}$

For each $a \in \mathbb{R}$

(B) $(\mathbb{R}^*, \cdot)$, $\mathbb{R}^* = \mathbb{R} \sim \{0\}$

(VI) $ab \in \mathbb{R}$ $\forall a, b \in \mathbb{R}$

Closure prop.

(III) $a+(b+c)=(a+b)+c$

$\forall a, b, c \in \mathbb{R}$

Associative prop.

(IV) Existence of additive identity.

$$a+\underbrace{0}_{\text{identity}}=0+a=a \quad \forall a \in \mathbb{R}$$

(X) Existance of multiplicative inverse.

For each $a \in \mathbb{R}^* = \mathbb{R} \sim \{0\}$

$\exists \; \frac{1}{a} \in \mathbb{R}^*$ s.t.

$$a \cdot \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \cdot a = 1$$

$\frac{1}{a} \downarrow a^{-1}$

(VII) $ab = ba, \ \forall a, b \in \mathbb{R}$

(VIII) $a(bc) = (ab)c, \ \forall a, b, c \in \mathbb{R}$

(IX) Existence of multiplicative identity

$$a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{R}$$

$1 \downarrow$ Identity.

Order axioms

O_1: For $a, b \in \mathbb{R}$, one and only one of the following holds.

(i) $a > b$ (ii) $a = b$ (iii) $b > a$

'Law of Trichotomy'.

(C) Distributive prop.

(X1) $a \cdot (b + c) = ab + ac$

(X11) $(a + b) \cdot c = a \cdot c + b \cdot c$

$\forall a, b, c \in \mathbb{R}$

Note (i) $(\mathbb{R}, +, \cdot)$ is a field.

(ii) $(\mathbb{Q}, +, \cdot)$ is a field.

Note: $(\mathbb{R}, +, \cdot)$ — ordered field.

$(\mathbb{Q}, +, \cdot)$ — ordered field.

Intervals:

$\boxed{\text{closed}}$ $[a, b]$

$= \{x \in \mathbb{R} : a \leq x \leq b\}$

$\boxed{\text{open}}$ $]a, b[$ or (a, b)

$= \{x \in \mathbb{R} : a < x < b\}$

O_2: For $a, b, c \in \mathbb{R}$

$a > b$ & $b > c \Rightarrow a > c$

O_3: for $a, b, c \in \mathbb{R}$

$a > b \Rightarrow a + c > b + c$

O_4: for $a, b, c \in \mathbb{R}$

$a > b$ & $c > 0 \Rightarrow ac > bc$

Bounds of a set

Bounded below set:

A non-empty subset S of $\mathbb{R}$ is said to be bounded below if $\exists$ a real no. k s.t.

$$k \leq x \quad \forall x \in S$$

Semi open, semi closed

$]a, b]$ or $(a, b]$

$= \{x \in \mathbb{R} : a < x \leq b\}$

$[a, b[$ or $[a, b)$

$= \{x \in \mathbb{R} : a \leq x < b\}$

Ex: ① $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$.

$x < 0 \quad \forall x \in \mathbb{R}^-$

bdd above.

(ii) $\mathbb{R}$ — Not bdd above

Ex: ① $\mathbb{N}$ — bdd below.

$1 \le x \quad \forall x \in \mathbb{N}$.

(ii) $\mathbb{R}$ — Not bounded below.

Bounded above set: A non-empty set $S \subseteq \mathbb{R}$ is said to be bounded above if $\exists$ a real no. K s.t

$x \le K \; \forall x \in S$

or

$\exists$ a +ve real no. M
s.t.

$$|x| \le M, \ \forall x \in S$$

i.e. $-M \le x \le M, \ \forall x \in S$.

<u>Ex:</u> $S =]0,1[$

$0 < x < 1 \ \forall x \in S$

Bdd.

<u>Bounded set</u>: A non-empty set $S \subseteq \mathbb{R}$ is bounded if it is bounded above as well as below.

or

A non-empty set $S \subseteq \mathbb{R}$ is bdd if $\exists$ real numbers k and K s.t.

$$k \le x \le K, \ \forall x \in S$$