(11) Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, over the complex number field.

$\boxed{\text{Ans: } \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}}$

IAS 2021
10M

(12) Show that $S = \{(x, 2y, 3x) : x, y \text{ are reals}\}$ is a subspace of $\mathbb{R}^3$. Find two bases of S. Also find the dimension of S.

Ans: $\{(1,0,3), (0,2,0)\}$
$\{(1,0,3), (0,1,0)\}$

IAS 2021
15M

(13) Find the matrix associated with the linear operator on $V_3(\mathbb{R})$ defined by $T(a,b,c) = (a+b, a-b, 2c)$ with respect to the ordered basis $B = \{(0,1,1), (1,0,1), (1,1,0)\}$

Ans: $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

IAS 2021
10M

(14) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2,1) = (5,7)$ and $T(1,2) = (3,3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2 then find the rank of A.

$\boxed{\text{Ans: Rank of } A = 2}$

IAS 2019
10M

Hints: $\{(2,1),(1,2)\}$ is a basis of $\mathbb{R}^2$.

Let $(a,b) \in \mathbb{R}^2$

$$(a,b) = x(2,1) + y(1,2)$$

$$y = \frac{2b-a}{3}, \quad x = \frac{2a-b}{3}$$

$$T(a,b) = \left[\frac{7a+b}{3}, \frac{11a-b}{3}\right]$$

$$A = \begin{bmatrix} 7/3 & \frac{1}{3} \\ 11/3 & -\frac{1}{3} \end{bmatrix}$$

(15) Consider the matrix mapping $A: \mathbb{R}^4 \to \mathbb{R}^3$. Where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and dimension of the image and those of the kernel of A.

IAS 2017
15M

Hints: $T(X) = AX$,

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$$

$$T(X) = x(1,1,3) + y(2,3,8) + z(3,5,13) + w(1,-2,-3)$$

Consider $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ By E-Row transf.

Basis of image of $A = \{(1,1,3), (0,1,2)\}$

Dim (image of A) = 2

Let $(x, y, z, w) \in \ker A$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By E-row trans.

Rank of coeff. matrix $= 2$

$\therefore 4-2 = 2$ free variable. (2 L.I. soln.)

Let $z = k_1$, $w = k_2$

$x + y + 3k_1 + k_2 = 0$

$y + 2k_1 - 3k_2 = 0 \Rightarrow y = -2k_1 + 3k_2$

$\Rightarrow x = -3k_1 - k_2 + 2k_1 - 3k_2$

$= -k_1 - 4k_2$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -k_1 - 4k_2 \\ -2k_1 + 3k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Basis of kernel $A = \{(-4, 3, 0, 1), (-1, -2, 1, 0)\}$

Dimension of $\ker A = 2$.

(16)

Prob! If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a L.T. $T: P_2(x) \to P_2(x)$ w.r.t. the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$, then find T.

IAS 2016
18 M

Hint:

$T(1-x) = 1\cdot 1 - 2(1+x) + 1\cdot(1+x^2)$ — (i)

$T(x(1-x)) = -1\cdot 1 + 1\cdot(1+x) + 2\cdot(1+x^2)$ — (ii)

$T(x\cdot(1+x)) = 2\cdot 1 - 1\cdot(1+x) + 3(1+x^2)$ — (iii)

T is Linear.

(i) $\Rightarrow$

$T(1) - T(x) = -2x + x^2$ — (iv)

(ii) $\Rightarrow$ $T(x) - T(x^2) = x^2 + x + 2$ — (v)

$T(x) + T(x^2) = 3x^2 - x + 4$ — (v.i)

(v) + (vi) will give $T(x) = 2x^2 + 3$

(iv) 〃 $T(1) = 3x^2 - 2x + 3$

(vi) 〃 $T(x^2) = x^2 - x + 1$

$\therefore T(ax^2 + bx + c) = aT(x^2) + bT(x) + c\cdot T(1)$

$= (a + 2b + 3c)x^2 + (-a - 2c)x + (a + 3b + 3c)$

(17) If $T: P_2(x) \to P_3(x)$ is such that $T(f(x)) = f(x) + 5\int_0^x f(t)\,dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ resp, find the matrix of T.

IAS 2016
6 M

Ans:
$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 6 & 5 \\ 0 & 5/2 & -1 \\ 0 & 0 & -5/3 \end{bmatrix}$$

(18) If $M_2(\mathbb{R})$ is a space of real matrices of order 2×2 and $P_2(x)$ is the space of real

Polynomials of degree at most 2, then find the matrix representation of $T: M_2(\mathbb{R}) \to P_2(x)$, such that:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$$

w.r.t. the standard bases of $M_2(\mathbb{R})$ and $P_2(x)$. Further find the null space of T.

IAS. 2016

10M

Ans: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$$N(T) = \left\{ k\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} : k \in \mathbb{R} \right\}$$

(19)

Prob: Let P_n denote the vector space of all real Polynomials of degree at most n and $T: P_2 \to P_3$ be a linear transformation given by

$$T(p(x)) = \int_0^x p(x)\,dx, \quad p(x) \in P_2$$

Find the matrix of T w.r.t. the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 resp. Also find the null space of T.

IAS 2013

10M

Ans: $\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

$$N(T) = \{0\}.$$