Maths Optional

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$$\lim_{t\to\infty}\int_1^t \frac{dx}{1+x^2} = \lim_{t\to\infty}\left[\tan^{-1}x\right]_1^t$$

$$= \lim_{t\to\infty}\left[\tan^{-1}t - \tan^{-1}1\right]$$

$$= \lim_{t\to\infty}\left[\tan^{-1}t - \pi/4\right]$$

$$= \pi/2 - \pi/4$$

$$= \pi/4 \text{ (finite)}$$

Integral converges.

$$\int_a^\infty f(x)\,dx$$

$$\lim_{t\to\infty}\int_a^t f(x)\,dx = \text{finite}$$

(converge)

Ex: $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$

Prob: Show that the series

$\sum \frac{1}{n^p}$ converges if $p>1$

and diverges if $0<p\le 1$.

$\rightarrow$ Let $u(x) = \frac{1}{x^p}$, $x \geq 1$

$u(x)$ is non-negative, monotonically decreasing and integrable fn.

$u(n) = \frac{1}{n^p} = u_n \quad \forall n \in \mathbb{N}$.

Cauchy's integral test:

If u is a non-negative, monotonically decreasing and integrable function such that $u(n) = u_n \quad \forall n \in \mathbb{N}$ then the series $\sum_{n=1}^{\infty} u_n$ and the integral $\int^{\infty} u(x)\,dx$ will converge or diverge together.

$$= \begin{cases} \left[\dfrac{x^{1-p}}{1-p} \right]_1^t, & p \neq 1 \\ \left[\log x \right]_1^t, & p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-p} \left[t^{1-p} - 1 \right], & p \neq 1 \\ \log t, & p = 1 \end{cases}$$

$$\lim_{t \to \infty} \int_1^t u(x)\,dx = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p \leq 1 \end{cases}$$

$\therefore$ By Integral test,

$\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$

will converge or diverge together.

$$\int_1^t u(x)\,dx = \int_1^t x^{-p}\,dx$$

prob: The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$,

$p > 0$ Converges for $p > 1$

and diverges for $p \leq 1$.

$\rightarrow$ Let $u(x) = \frac{1}{x(\log x)^p}$

For $x \geq 2$, $u(x)$ is non-negative, monotonically decreasing and integrable fn.

Also $u(n) = u_n = \frac{1}{n(\log n)^p}$ $\forall n \geq 2$

$\therefore \int_{1}^{\infty} u(x)\,dx$ converges for $p > 1$

" diverges for $p \leq 1$

$\therefore \sum u_n$ conv. for $p > 1$

" div. for $p \leq 1$

$$= \begin{cases} \left[\dfrac{z^{1-p}}{1-p}\right]_{\log 2}^{\log t}, & p \neq 1 \\ \left[\log z\right]_{\log 2}^{\log t}, & p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-p}\left[(\log t)^{1-p} - (\log 2)^{1-p}\right], & p \neq 1 \\ \log(\log t) - \log(\log 2), & p = 1 \end{cases}$$

$$\lim_{t\to\infty} \int_2^t \frac{dx}{x(\log x)^p} = \begin{cases} \dfrac{(\log 2)^{1-p}}{p-1}, & p > 1 \\ \infty, & p \leq 1 \end{cases}$$

$\therefore$ By integral test

$\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} u(x)\,dx$ will

Converge or diverge tog.

$$\int_2^t \frac{dx}{x(\log x)^p}, \quad \text{let } z = \log x$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$dz = \frac{1}{x}\,dx$$

$$= \int_{\log 2}^{\log t} \frac{dz}{z^p}$$

Prob: using integral test, find the nature of the series. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$\longrightarrow$ $u(x) = \frac{1}{x^2+x}$

For $x \geq 1$, $u(x)$ is non-negative, monotonically decreasing and integrable fn!

$\therefore \int_2^{\infty} \frac{dx}{x(\log x)^p}$ Converges for $p>1$

" div for $p \leq 1$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ conv. for $p>1$

" div. for $p \leq 1$.

$$= \int_1^t \left[\frac{1}{x} - \frac{1}{x+1}\right] dx$$

$$= \left[\log x - \log(x+1)\right]_1^t$$

$$= \left[\log t - \log(1+t) + \log 2\right]$$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x(x+1)} = \lim_{t\to\infty}\left[\log\frac{t}{1+t} + \log 2\right]$$

$$= \lim_{t\to\infty}\left[\log\frac{1}{1+\frac{1}{t}} + \log 2\right]$$

$$= \log 2$$

Also:

$$u(n) = u_n = \frac{1}{n^2+n}$$

$\forall n \in \mathbb{N}$.

By integral test:

$\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\, dx$ will have same nature.

$$\int_1^t \frac{1}{x(x+1)}\, dx$$

Gauss's test : If $\sum u_n$ is a +ve term series such that

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{r_n}{n^p},$$

where $\alpha > 0$, $p > 1$ and $\langle r_n \rangle$ is a bdd seq. then

(I) For $\alpha \neq 1$, $\sum u_n$ converge for $\alpha > 1$ and div. for $\alpha < 1$ whatever β may be.

(II) For $\alpha = 1$, $\sum u_n$ converge for

$\therefore \int_1^\infty \frac{dx}{x(x+1)}$ converge.

$\therefore \sum_{n=1}^\infty \frac{1}{n^2+n}$ converge where $\alpha, \beta, \gamma, \delta$ - - independent of n. then

(1) can be written as.

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{1}{n^2}\left[\gamma + \frac{\delta}{n} + \cdots\right]$$

$$= \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^2}.$$

$$\gamma_n = \gamma + \frac{\delta}{n} + \cdots$$

$\lim \gamma_n = \gamma$

$\beta > 1$ and diverges for $\beta \leq 1$.

Remarks (1) Gauss's test is stronger than Ratio test and Raabe's test also.

(ii) $$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2} + \frac{\delta}{n^3} + \cdots \quad (1)$$

Prob: Test for the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

$$\rightarrow \quad u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}$$

$$u_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2 (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2 (2n+3)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}$$

$\therefore \langle r_n \rangle$ is cgt.

$\therefore$ gt is bdd.

Thus for application of Gauss's test. expand $\frac{u_n}{u_{n+1}}$ in the powers of $\frac{1}{n}$.

Here $\alpha = 1$, $\beta = 1$

$\therefore$ By Gauss's test

$\sum u_n$ diverge.

$\therefore \dfrac{u_n}{u_{n+1}} = \dfrac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$

$\dfrac{u_n}{u_{n+1}} = \left(1+\frac{3}{2n}\right)^2 \left(1+\frac{1}{n}\right)^{-2}$

$= \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left[1-\frac{2}{n}+\frac{3}{n^2} - \cdots\right]$

$= 1+\frac{1}{n}(-2+3)+\frac{1}{n^2}\left(3-6+\frac{9}{4}\right)+$

$= 1+\frac{1}{n}-\frac{3}{4n^2}+\cdots$ higher power of $\frac{1}{n}$

By Ratio test

$\sum u_n$ converges for $\frac{1}{x} > 1$

i.e. $x < 1$

" div. for $\frac{1}{x} < 1$

i.e. $x > 1$

Test fails when $x = 1$.

For $x = 1$, Applying Gauss's test

Prob: Test for convergence of the series $\sum \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2} \cdot x^{n-1}$, $x > 0$

$\rightarrow \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x}$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

$$= \left[1 + \frac{1}{n} - \frac{1}{4n^2} \cdot \text{---}\right]$$

$\alpha = 1, \quad \beta = 1$

$\therefore$ By Gauss's test, $\sum u_n$ diverge.

$\sum u_n$ conv. for $x < 1$

" div. " $x \geq 1$.

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left[1 + \frac{2}{n} + \frac{1}{n^2}\right]\left[1 - \frac{1}{n} + \frac{3}{4n^2} + \cdots\right]$$

$$= \left[1 + \frac{1}{n}(-1+2) + \frac{1}{n^2}\left(\frac{3}{4} - 2 + 1\right) + \cdots + \text{higher power of } \frac{1}{n}\right]$$

(i) $\alpha \neq 1$ $\quad \sum u_n$ conv. for

$\frac{1}{x} > 1$ i.e. $x < 1$

div. $\frac{1}{x} < 1$

i.e. $x > 1$

(ii) $\alpha = 1$, i.e. $x = 1$

$\beta = \frac{1}{1} = 1$

$\therefore \sum u_n$ div.

Or

Directly applying Gauss's test

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{1}{n} - \frac{1}{4n^2} - - \right]$$

$$= \left[\frac{1}{x} + \frac{\frac{1}{x}}{n} - \frac{\frac{1}{x}}{4n^2} - - \right]$$

$\alpha = \frac{1}{x}$, $\quad \beta = \frac{1}{x}$

HW prob: Test for convergence of the series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta \cdot (\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} x^2$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta \cdot (\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot (\gamma+1) \cdot (\gamma+2)} \cdot x^3$$

$$+ \cdots$$