KGS IAS

Maths Optional

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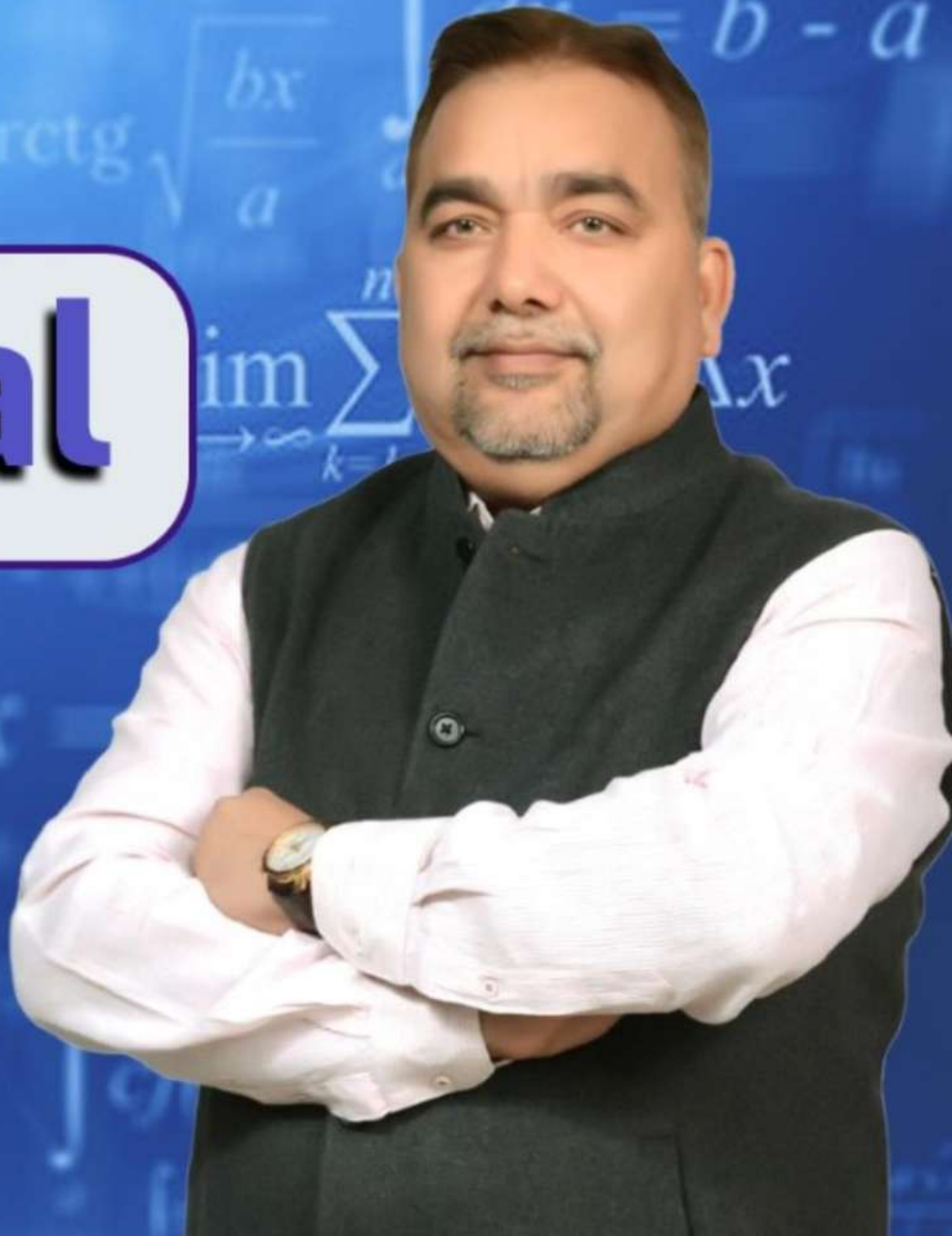

$$\lim \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n+2}{\left(1+\frac{1}{n}\right)^2}$$

$$= \infty > 1$$

$\therefore$ By Ratio Test

$\sum |u_n|$ is dgt.

$\therefore \sum u_n$ converges abs.

so is dgt.

HW $\sum (-1)^{n-1} \cdot \frac{n^2}{\lfloor n+1}$

Let $u_n = (-1)^{n-1} \cdot \frac{n^2}{\lfloor n+1}$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n^2}{\cancel{\lfloor n+1}} \times \frac{\cancel{\lfloor n+2}^{\,n+2}}{(n+1)^2}$$

$$= \frac{n+2}{\left(\frac{n+1}{n}\right)^2}$$

Prob: Test for convergence of the series

$$\sum \frac{(-1)^{n-1}}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)$$

$\rightarrow$ Let $u_n = \dfrac{(-1)^{n-1}}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$

$$|u_n| = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{n+1-n+1}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$$

$$= \frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)} \sim \frac{\cancel{2}}{\cancel{2}\, n\cdot\sqrt{n}}$$

$$= \frac{1}{n^{\frac{3}{2}}}$$

Let $v_n = \dfrac{1}{n^{\frac{3}{2}}}$

$$\lim \frac{|u_n|}{v_n} = \lim \frac{2\,\cancel{n^{\frac{3}{2}}}}{\cancel{n\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

$$= 1 \neq 0$$

Prob: Test for conv. and absolute convergence of the series $\sum \frac{(-1)^{n+1}}{\log(n+1)}$.

$\rightarrow \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} \cdots -$

Let $u_n = \frac{1}{\log(n+1)}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0$$

By Limit form of comparison test.

$\sum |u_n|$ converges

as $\sum \frac{1}{n^{3/2}}$ conv. $\left(p = \frac{3}{2} > 1\right)$

$\therefore \sum u_n$ conv. absolutely.

$\therefore \sum u_n$ conv.

$\therefore$ By Leibnitz's test the given series is cgt.

For check absolute conv. consider the series

$$\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{\log}{\log 4} + \cdots$$

$\log n < n \quad \forall\, n \geq 2$

$\Rightarrow \frac{1}{\log n} > \frac{1}{n}$

$\because \log x$ is $\uparrow \quad \forall\, x > 0$

$n+2 > n+1$

$\Rightarrow \log(n+2) > \log(n+1)$

$\Rightarrow \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \quad \forall\, n \geq 1$

$\Rightarrow u_{n+1} < u_n$

i.e. $u_n > u_{n+1} \; \forall n.$

Prob: Test the convergence and absolute convergence of the series

$$\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \cdots$$

$\rightarrow$ Let $u_n = \dfrac{1}{(n+1)\log(n+1)}$

$$\lim u_n = \lim \frac{1}{(n+1)\log(n+1)} = 0$$

i.e. $\frac{1}{n} < \frac{1}{\log n} \quad \forall\, n \geq 2$

By comparison test.

$\sum_{n=2}^{\infty} \frac{1}{\log n}$ is divergent

as $\sum \frac{1}{n}$ is dgt.

$\therefore$ The given series is not abs. cgt.

$\therefore$ The given series is cond. cgt.

$$\Rightarrow \frac{1}{(n+2)\log(n+2)} < \frac{1}{(n+1)\log(n+1)}$$

$$\Rightarrow u_{n+1} < u_n \quad \underline{\forall n}.$$

$\therefore$ By Leibnitz's test.

the given series is cgt.

Now checking for abs. Convergence

$$\frac{1}{2\log 2} + \frac{1}{3\log 3} + \frac{1}{4\log 4} + \cdots —$$

Let $u(x) = \frac{1}{x\log x}$

$$u'(x) = \frac{-[\log x + 1]}{(x\log x)^2}$$

$$u'(x) < 0 \quad \forall\, x \geq 2$$

$\therefore u(x)$ is $\downarrow$ $\forall\, x \geq 2$

Now $n+2 > n+1$

$$\Rightarrow u(n+2) < u(n+1)$$

Prob: Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{\lfloor n}$

is abs. cgt.

$\rightarrow$ Let $u_n = (-1)^{n+1} \cdot \frac{2^n}{\lfloor n}$

$$\lim \frac{|u_n|}{|u_{n+1}|} = \lim \frac{\cancel{2^n}}{\cancel{\lfloor n}} \times \frac{\cancel{\lfloor n+1}^{(n+1)}}{\cancel{2^{n+1}}_{2}}$$

$$= \infty > 1$$

$$\left[\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \quad \begin{array}{l} \text{con. for } p>1 \\ \text{div. for } p \le 1 \end{array} \right]$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n \log n}$ div.

$\therefore$ The given series is not abs. cgt.

$\therefore$ The given series is cond. cgt.

and if $\sum u_n$ is a convergent series then the series $\sum b_n u_n$ is also convergent.

Cor : A convergent series $\sum u_n$ (need not converge absolutely) remains convergent if its terms are each multiplied by a factor a_n, where $\langle a_n \rangle$ is bdd & monotonic.

$\therefore$ Ratio test,

the given series conv. abs.

Tests for series of arb. terms

Convergent but not abs. Convergent.

Abel's test : If b_n is a +ve, monotonically decreasing function

$\langle a_n \rangle$ is a monotonically increasing and bdd seq.

$\therefore$ By Abel's test

$\sum a_n \cdot u_n$ is cgt.

$\therefore \sum \frac{(-1)^{n-1}}{n} \cdot \left(1+\frac{1}{n}\right)^n$ is cgt.

Prob: Test the convergence of

$$\sum \frac{(-1)^{n-1}}{n} \left(1+\frac{1}{n}\right)^n.$$

Let $u_n = \frac{(-1)^{n-1}}{n}$.

& let $a_n = \left(1+\frac{1}{n}\right)^n$.

$\sum u_n$ is cgt. by Leibnitz's test.

$\sum u_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots$

$\therefore$ The series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$= \sum \frac{1}{n^2}$$

is cgt.

$\therefore \sum u_n$ conv. abs.

$\therefore \sum u_n$ conv.

a_n is a +ve, monotonically $\downarrow$ fn.

Prob: Test the convergence of

$$1 - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 4^2} \cdots$$

$$= \sum \frac{(-1)^{n-1}}{n^2 \cdot (2n-1)}$$

$\rightarrow$ Let $u_n = \frac{(-1)^{n-1}}{n^2}$, $a_n = \frac{1}{2n-1}$

$\rightarrow$ Let $u_n = (n^3+1)^{\frac{1}{3}} - n$, $b_n = \frac{1}{\log n}$

Σu_n is cgt.

$\langle b_n \rangle$ is a +ve seq. and is mono. $\downarrow$ seq.

$\therefore$ By Abel's test

$\Sigma u_n b_n$ is cgt.

$\therefore$ By Abel's test

$\Sigma a_n \cdot u_n$

$= \Sigma \frac{(-1)^{n-1}}{n^2 \cdot (2n-1)}$ is cgt

Prob: Test the conv. of the series.

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{\frac{1}{3}} - n}{\log n}.$$

→ Let $\sum u_n = \sum (-1)^{n-1}$

Let $\langle s_n \rangle$ be its S.O.P.S.

$$s_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$\therefore \langle s_n \rangle$ is bdd.

$\sum u_n \cdot b_n$ reduces to $b_1 - b_2 + b_3 - -$

Thus we obtain, if b_n is +ve, monotonically decreasing

Dirichlet's test: If b_n is +ve, monotonically decreasing fn with limit zero and if for the series $\sum u_n$, the S.O.P.S. is bdd then the series $\sum u_n b_n$ is cgt.

Cor: Leibnitz's test is a particular case of Dirichlet's test.

Let $\langle s_n \rangle$ be the s.o.p.s of $\sum u_n$

$$s_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$\therefore \langle s_n \rangle$ is bdd.

Now, observe that

(I) $b_n > 0$ th.

(II) $n+1 > n$

$\Rightarrow (n+1)^p > n^p$, $p > 0$

function with limit zero

then $\sum u_n b_n$ converges.

i.e. $b_1 - b_2 + b_3 - b_4 - - $ conv.

<u>Prob</u>: Discuss the conv. of

$$\sum \frac{(-1)^{n-1}}{n^p}, \; p > 0.$$

$\rightarrow$ Let $u_n = (-1)^{n-1}$, $b_n = \dfrac{1}{n^p}$

Rearrangement (De-arrangement) of series:

- Cgt.
  - Cond. cgt. (cgt but not abs.)
  - Abs. cgt. (cgt)

$$\Rightarrow \frac{1}{(n+1)^p} < \frac{1}{n^p}$$

$$\Rightarrow b_{n+1} < b_n \quad \forall n.$$

(iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p}$

$= 0$, as $p>0$

$\therefore$ By Dirichlet's test.

$$\sum b_n u_n = \sum \frac{(-1)^{n-1}}{n^p} \text{ is cgt.}$$

$(p>0)$

Rearrangement of terms:

A series $\sum v_n$ is said to arise from a series $\sum u_n$ by a rearrangement of terms if $\exists$ a one-to-one correspondence (bijection) b/w the terms of the series, so that every term in the 1st series

Ex:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots \text{—} = \log 2$$

(Cond. cgt.)

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} \cdots \text{—}$$

$$= \frac{1}{2} \log 2$$

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} + \cdots$$

$$= \log 2.$$

occupies a perfectly definite
place in the second series
and conversely.