KGS IAS

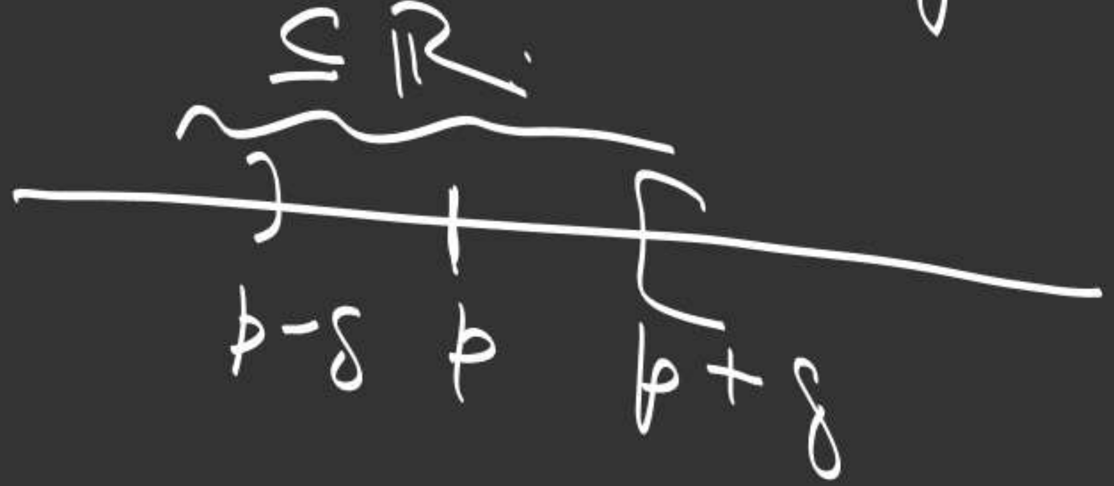
Maths Optional

By Dhrub Singh Sir

Ex: ① $\mathbb{R}$ — is a nbd of each of its pts.

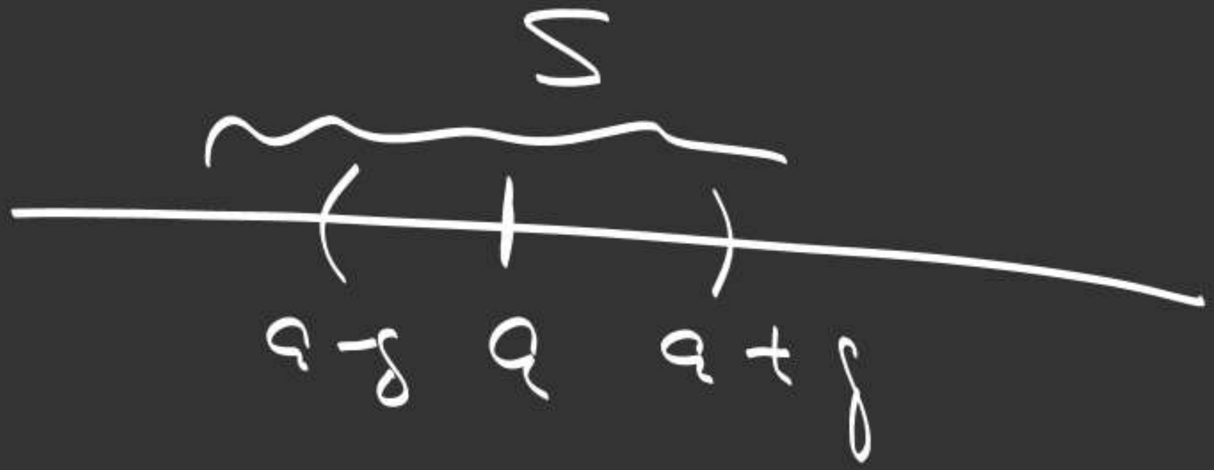
$$p \in \mathbb{R}$$

$$]p-\delta, p+\delta[\subseteq \mathbb{R} \text{ for any } \delta > 0.$$



Set as nbd: A set $S \subseteq \mathbb{R}$ is said to be a nbd of a pt. 'a' $\in \mathbb{R}$ iff $\exists$ some $\delta > 0$ s.t.

$$a \in \,]a-\delta, a+\delta[\, \subseteq S.$$



Ex ③ $\mathbb{N}, \mathbb{Z}, W$ -- not a nbd of its pts.

Result ① An open interval is a nbd of each of its point.

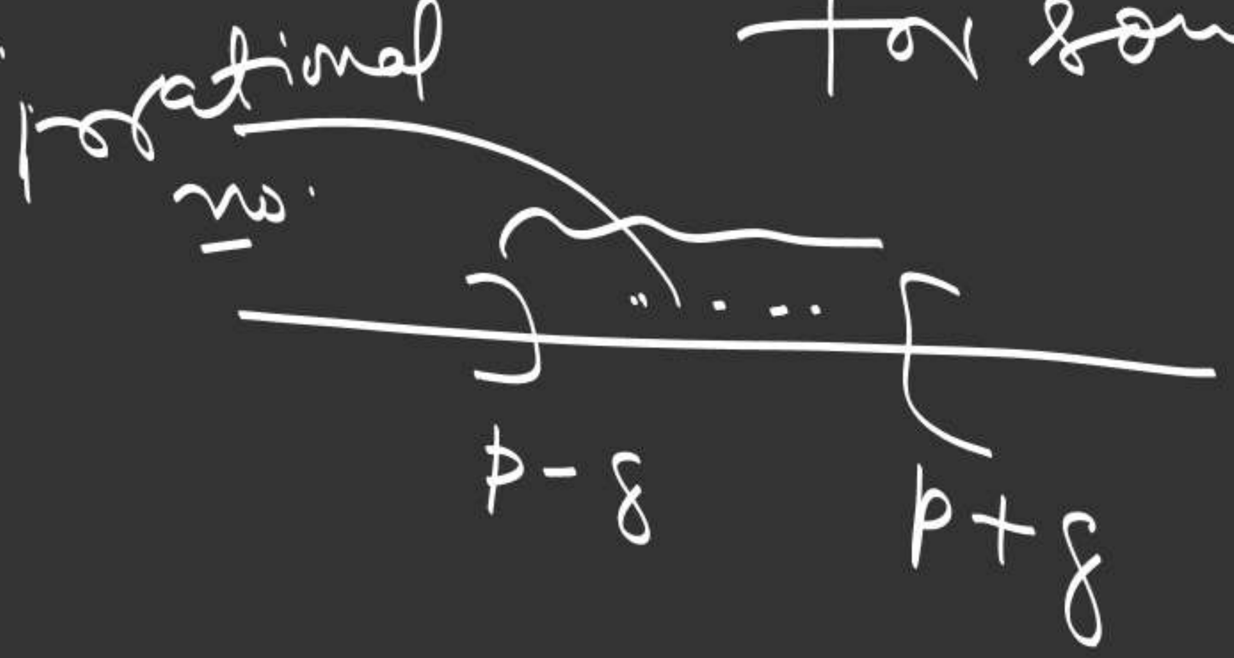
② $S=[a,b]$ — nbd of each of its point except at a & b.

Ex ② $\mathbb{Q}$ is not a nbd of any of its point.

$$p \in \mathbb{Q}$$

$$]p-\delta, p+\delta[\not\subseteq \mathbb{Q}$$

for some $\delta > 0$.



Result: Every superset of a nbd of a point is also a nbd. of that point.

Reason:

M is a nbd of 'p'.

$\exists$ some $\delta > 0$

s.t

$]p-\delta, p+\delta[\subseteq M \subseteq N$

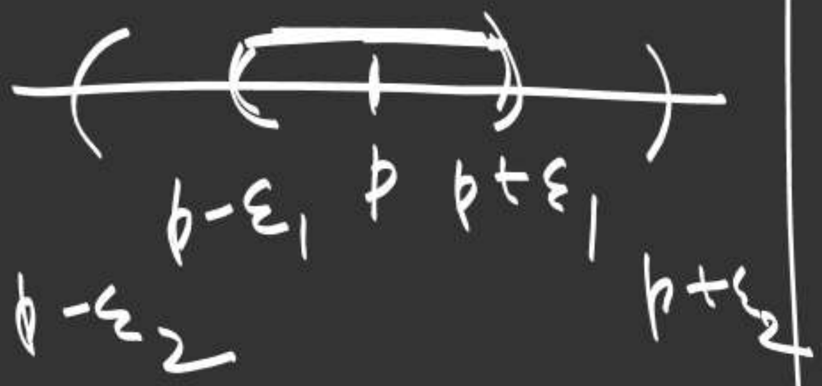
$\therefore$ N is a nbd of 'p'.

(3) A non-empty finite set can not be a nbd of any of its points.

(4) ϕ — is a nbd of each of its points.

Reason: There is no point in ϕ of which it is not a nbd.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$



$]p-\varepsilon, p+\varepsilon[\subseteq]p-\varepsilon_1, p+\varepsilon_1[\subseteq M$

$]p-\varepsilon, p+\varepsilon[\subseteq]p-\varepsilon_2, p+\varepsilon_2[\subseteq N$

$\therefore]p-\varepsilon, p+\varepsilon[\subseteq M \cap N$

$M \cap N$ — nbd of p.

<u>Result</u>: The intersection of two nbds of a point is also a nbd of that pt.

$\boxed{\text{Reason}}$: M, N — nbd of p.

$\exists$ some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ s.t.

$]p-\varepsilon_1, p+\varepsilon_1[\subseteq M$

$]p-\varepsilon_2, p+\varepsilon_2[\subseteq N$

i.e. $\exists$ some $\varepsilon > 0$ (may be very small)

s.t.

$p \in \,]p-\varepsilon, p+\varepsilon[\, \subseteq S$

Ex: (1) Every pt of $\mathbb{R}$ is an interior point.

(11) $[a, b]$ — All pts except a & b are interior point.

Result: M is a nbd of 'p' and N is also a nbd of p. then $M \cup N$ is also a nbd of 'p'.

Interior point of a set:

Let $S \subseteq \mathbb{R}$, $p \in S$ is called an interior pt. of S iff S is a nbd of 'p'.

Ex: (I) $\operatorname{int} \mathbb{R} = \mathbb{R}$

(II) $\operatorname{int} [a,b] =]a,b[$

(III) $\operatorname{int} [a,b[=]a,b[$

(IV) $\operatorname{int} \mathbb{N} = \phi$.

open set: A set $S \subseteq \mathbb{R}$ is said to be an open set if it is a nbd of each of its point.

(III) $\mathbb{N}$ —— has no interior point.

Interior of a set: The set of all interior points of a set S is called interior of the set.

Notation: $\operatorname{int} S$ or S°

(ii) $[a, b]$ — not open

$\text{Int}\,[a, b] =]a, b[$

$\text{Int}\,[a, b] \neq [a, b]$

$]a, b]$ — not open

$[a, b[$ — not open

(iii) $\mathbb{R}$ — open

$\text{Int}\,\mathbb{R} = \mathbb{R}$

(iv) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \sim \mathbb{Q}$ — not open

i.e., for each $p \in S$,

$\exists$ some $\varepsilon > 0$ s.t.

$p \in]p - \varepsilon, p + \varepsilon[\subseteq S$

Note: S is open $\iff \text{Int}\,S = S$

Ex (1) Every open interval is an open set.

$\text{Int}\,]a, b[=]a, b[$

To prove: $U_1 \cup U_2$ is open.

Let $p \in U_1 \cup U_2$

$\Rightarrow p \in U_1$ or $p \in U_2$

Case (I) $p \in U_1$

$\because U_1$ is open.

$\therefore \exists$ some $\varepsilon_1 > 0$ s.t.

$p \in \,]p-\varepsilon_1, p+\varepsilon_1[\, \subseteq U_1 \subseteq U_1 \cup U_2$

(I)

Case (II) $p \in U_2$

(V) $\mathbb{R}^+ = \,]0, \infty[$

open set.

(VI) ϕ — open

Th: union of two open set is open.

Proof: Let U_1 and U_2 be open sets.

Th: The union of an arbitrary family of open sets is an open set.

proof: let $\mathcal{F} = \{G_\lambda : \lambda \in \Lambda\}$ ($\Lambda \to$ index set) be the arbitrary family of open set.

To prove: $\bigcup_{\lambda \in \Lambda} G_\lambda$ is an open set.

G_2 is open

$\therefore \exists$ some $\varepsilon_2 > 0$ s.t.

$p \in \,]p - \varepsilon_2, p + \varepsilon_2[\, \subseteq G_2 \subseteq G_1 \cup G_2$ — (ii)

From (i) & (ii)

$G_1 \cup G_2$ is a nbd of p.

$\because p$ is any point of $G_1 \cup G_2$

$\therefore G_1 \cup G_2$ is a nbd of each of its points. $\therefore G_1 \cup G_2$ is open.

$\because$ p is any point of $\bigcup_{\lambda\in\Delta} G_\lambda$.

$\therefore$ $\bigcup_{\lambda\in\Delta} G_\lambda$ is a nbd of each of its points.

$\therefore$ $\bigcup_{\lambda\in\Delta} G_\lambda$ is an open set.

Let $p \in \bigcup_{\lambda\in\Delta} G_\lambda$

$\therefore$ $\exists$ some $\alpha \in \Delta$ s.t.

$p \in G_\alpha$.

$\because$ G_α is open.

$\therefore$ $\exists$ some $\varepsilon > 0$ s.t.

$p \in]p-\varepsilon, p+\varepsilon[\subseteq G_\alpha \subseteq \bigcup_{\lambda\in\Delta} G_\lambda$

$\therefore$ $\bigcup_{\lambda\in\Delta} G_\lambda$ is a nbd of p.

Th: The intersection of two open set is open.

Proof: Let u_1 & u_2 be open set.

To prove: $u_1 \cap u_2$ is open.

Let $p \in u_1 \cap u_2$

$\Rightarrow p \in u_1$ and $p \in u_2$

$\because$ u_1 and u_2 are open set.

$\therefore$ $\exists$ some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ s.t.

$$p \in \left]p-\varepsilon_1, p+\varepsilon_1\right[\subseteq u_1 \quad \text{— (i)}$$

$$p \in \left]p-\varepsilon_2, p+\varepsilon_2\right[\subseteq u_2 \quad \text{— (ii)}$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

$$\therefore \left]p-\varepsilon, p+\varepsilon\right[\subseteq \left]p-\varepsilon_1, p+\varepsilon_1\right[\subseteq u_1$$

$$\left]p-\varepsilon, p+\varepsilon\right[\subseteq \left]p-\varepsilon_2, p+\varepsilon_2\right[\subseteq u_2$$

Note ① The intersection of finite number of open sets is open.

② The intersection of an infinite collection of open sets need not be an open set.

Ex①: $G_n = \left]-\frac{1}{n}, \frac{1}{n}\right[$

$G_1 =]-1, 1[$, $G_2 = \left]-\frac{1}{2}, \frac{1}{2}\right[$ - - are open sets.

$\therefore p \in]p-\varepsilon, p+\varepsilon[\subseteq G_1 \cap G_2$

$\therefore G_1 \cap G_2$ is a nbd of p.

And p is any point of $G_1 \cap G_2$

$\therefore G_1 \cap G_2$ is a nbd of each of its point.

So, $G_1 \cap G_2$ is open

$G_1 =]0,1[,\ G_2 =]0,2[,\ G_3 =]0,3[$

$\cdots$

$G_1 \cap G_2 \cap G_3 \cap G_4 \cap \cdots$

$=]0,1[\cap]0,2[\cap]0,3[\cap \cdots$

$=]0,1[$ —— open set.

$G_1 \cap G_2 \cap G_3 \cap G_4 \cdots$

$=]-1,1[\cap]-\frac{1}{2},\frac{1}{2}[\cap]-\frac{1}{3},\frac{1}{3}[\cap \cdots$

$= \{0\}$ —— not open

(ii) $G_n =]0,n[,\ n \in \mathbb{N}$

Each G_n is an open set.

Limit point of a set: A point $p \in \mathbb{R}$ is said to be a limit point of a set $S \subseteq \mathbb{R}$ if every nbd of 'p' contains a point of S different from p.

$\rightarrow$ p is a limit point of S if

$$]p-\varepsilon, p+\varepsilon[\cap S \sim \{p\} \neq \phi$$

for each $\varepsilon > 0$.

(3) Every open interval is an open set but every open set need not be an open interval.

Ex: $G =]0,1[\cup]2,3[$

$\downarrow$

open set

G is not an interal

$p \in \mathbb{R}$

$]p-\varepsilon, p+\varepsilon[\cap \mathbb{R}$ = infinite set

for each $\varepsilon > 0$

Note ① cluster point or condensation point or accumulation point.

$\rightarrow$ p is a limit point of $S \subseteq \mathbb{R}$

if every nbd. of p contains infinitely many points of S.

i.e. $]p-\varepsilon, p+\varepsilon[\cap S$ = infinite set

for each $\varepsilon > 0$

Ex ① Every point of $\mathbb{R}$ is a limit point of $\mathbb{R}$.

(3) Number of limit points of a set = 0 or 1 or finite or infinite.

(4) p is not a limit point of S if we can find some $\varepsilon > 0$ s.t.

$$]p-\varepsilon, p+\varepsilon[\cap S = \phi \text{ or } \{p\}.$$

(2) limit point of a set may or may not belong to the set.

Ex: $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

0 is a limit pt. of S.

$0 \notin S$.

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

$\varepsilon = \frac{1}{2}$

$1 - \frac{1}{2} = 0.5 \qquad 1 + \frac{1}{2} = 1.5$

$$(0.5, 1.5) \cap \mathbb{N} = \{1\}.$$