KGS IAS

Maths Optional

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and $x_n > c$; converging to 'c', the seq. $\langle f(x_n) \rangle \to l$.

(2) Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of $A \cap]-\infty, c[$ then the following st. are equivalent:

(i) $\lim_{x \to c-} f(x) = l$

Sequential criterion for one sided limits

(1) Let $A \subseteq \mathbb{R}$, $f: A \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of $A \cap]c, \infty[$ then the following statements are equivalent:

(i) $\lim_{x \to c+} f(x) = l$

(ii) For every seq $\langle x_n \rangle$, $x_n \in A$

$$\lim_{x\to 0-} \operatorname{sgn}(x) = \lim_{\substack{x\to 0\\ x<0}} \operatorname{sgn}(x)$$

$$= \lim_{\substack{x\to 0\\ x<0}} -1 = -1$$

$$\lim_{x\to 0+} \operatorname{sgn}(x) = \lim_{\substack{x\to 0\\ x>0}} \operatorname{sgn}(x)$$

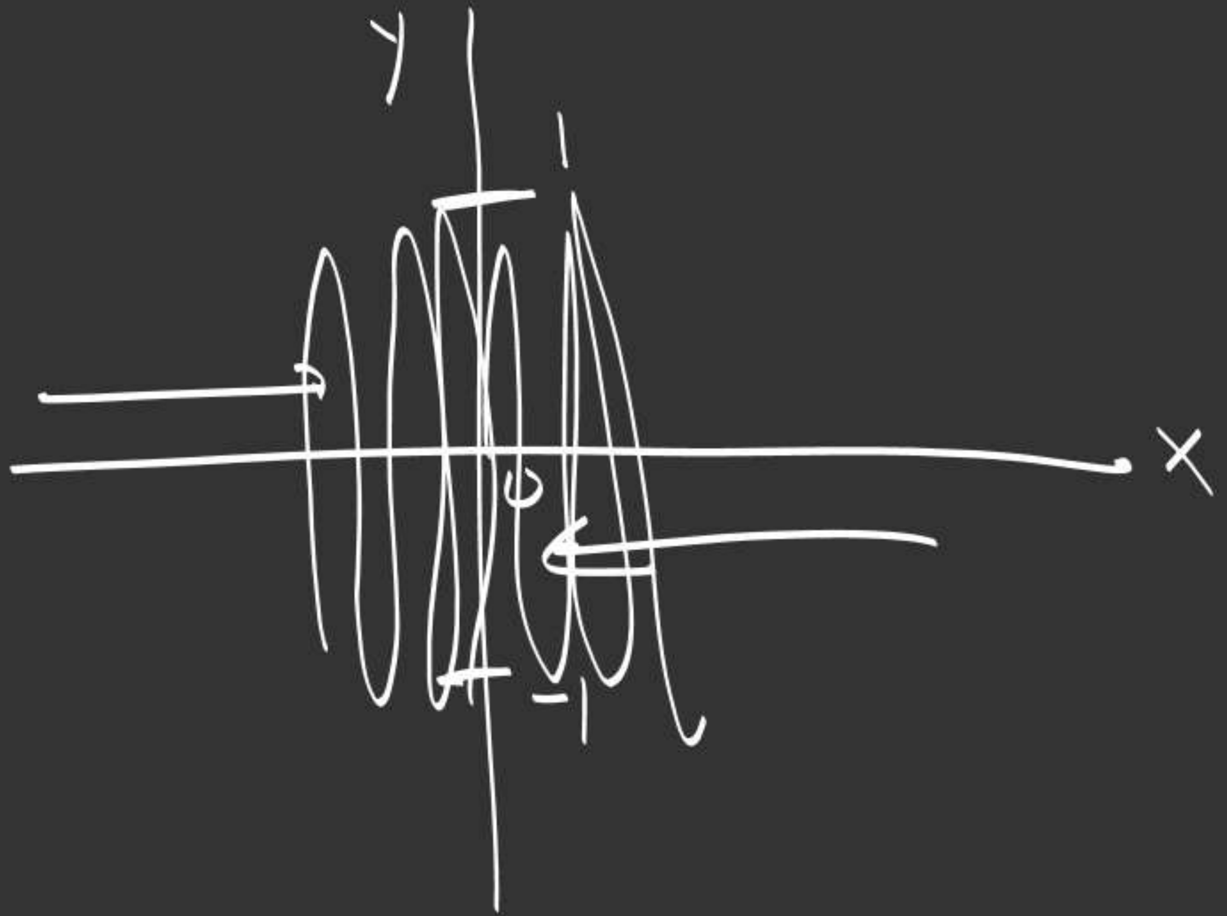
$$= \lim_{\substack{x\to 0\\ x>0}} 1 = 1$$

$$\therefore \lim_{x\to 0-} \operatorname{sgn}(x) \neq \lim_{x\to 0+} \operatorname{sgn}(x)$$

(ii) For every seq. $\langle x_n \rangle$, $x_n \in A$, $x_n < c$; converging to 'c', the seq. $\langle f(x_n) \rangle \to l$

Prob: Does $\lim_{x\to 0} \operatorname{sgn}(x)$ exist?

$\to$ $$\operatorname{sgn}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$$



$$\lim_{x\to 0^+} \sin\frac{1}{x} = \lim_{\substack{x\to 0 \\ x>0}} \sin\frac{1}{x}$$

= A no. fluctuating b/w -1 & 1.

$\therefore \lim_{x\to 0} \mathrm{Sgn}(x)$ does not exist.

<u>prob</u>: Does $\lim_{x\to 0} \sin\frac{1}{x}$ exist?

$$\to \lim_{x\to 0^-} \sin\frac{1}{x}$$

$$= \lim_{\substack{x\to 0 \\ x<0}} \sin\frac{1}{x}$$

= A no. fluctuating b/w -1 & 1.

$$\lim_{x\to 0+} x\sin\frac{1}{x} = \lim_{\substack{x\to 0\\ x>0}} x\sin\frac{1}{x}$$

$$= 0 \times \text{A finite value}$$

$$= 0$$

$$\therefore \lim_{x\to 0-} x\sin\frac{1}{x} = \lim_{x\to 0+} x\sin\frac{1}{x}$$

$$= 0$$

$\therefore \lim_{x\to 0} x\sin\frac{1}{x}$ exist.

$\therefore \lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Prob: Show that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

$$\to \lim_{x\to 0-} x\sin\frac{1}{x}$$

$$= \lim_{\substack{x\to 0\\ x<0}} x\sin\frac{1}{x}$$

$$= 0 \times \text{A finite value.}$$

$$= 0$$

(ii) $\lim_{x \to c} f(x) = -\infty$

$f(x) < -G$ whenever $0 < |x-c| < \delta$

(iii) $\lim_{x \to \infty} f(x) = l$

For any $\varepsilon > 0$ $\exists$ some +ve real no G (May be very large & depending on ε)

Infinite limits and limits at infinity :

(i) $\lim_{x \to c} f(x) = +\infty$

Let G be a +ve real no. (May be very large) $\exists$ a $\delta > 0$ (depending on G) s.t

$f(x) > G$ whenever $0 < |x-c| < \delta$

be very large) $\exists$ a +ve real no. G' s.t.

$$f(x) > G \text{ whenever } x > G'.$$

(VI) $\lim_{x \to \infty} f(x) = -\infty$

$$f(x) < -G \text{ whenever } x > G'$$

(VII) $\lim_{x \to -\infty} f(x) = \infty$

$$f(x) > G \text{ whenever } x < -G'.$$

$$|f(x) - l| < \varepsilon \text{ whenever } x > G$$

(IV) $\lim_{x \to -\infty} f(x) = l$

$$|f(x) - l| < \varepsilon \text{ whenever } x < -G$$

(V) $\lim_{x \to \infty} f(x) = \infty$

For any +ve real no. G (May

$\therefore \lim_{x \to \infty} \frac{1}{x} = 0$

Prob: Show that $\lim_{x \to -\infty} \frac{1}{x} = 0$

$\rightarrow$ Let $\varepsilon > 0$ be given

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{|x|}$$

$$\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \varepsilon \text{ if } |x| > \frac{1}{\varepsilon}$$

i.e. $x < -\frac{1}{\varepsilon}$ or $x > \frac{1}{\varepsilon}$.

Prob: $\lim_{x \to \infty} \frac{1}{x} = 0$

$\rightarrow$ Let $\varepsilon > 0$ be given

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \varepsilon$$

if $x > \frac{1}{\varepsilon}$

Let G be a real no. $> \frac{1}{\varepsilon}$

we have

$\left|\frac{1}{x} - 0\right| < \varepsilon$ whenever $x > G$.

(iii) $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

→ (i) Let $G > 0$ be a real no. (May be very large)

$$\frac{1}{x} > G \text{ if } x < \frac{1}{G}.$$

Let $\delta > 0$ be a no. less than $\frac{1}{G}$.

$\therefore \frac{1}{x} > G$ if $0 < x < \delta$

Let M be a +ve no. $> \frac{1}{\varepsilon}$

$\therefore \left|\frac{1}{x} - 0\right| < \varepsilon$ whenever $x < -M$.

Prob: Show that

(i) $\lim_{x \to 0+} \frac{1}{x} = +\infty$

(ii) $\lim_{x \to 0-} \frac{1}{x} = -\infty$

Prob: Evaluate $\lim_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$.

$\rightarrow$ As $\lim_{x\to 0-} \frac{1}{x} = -\infty$

$\therefore \lim_{x\to 0-} e^{1/x} = 0.$

As $\lim_{x\to 0+} \frac{1}{x} = +\infty$

$\therefore \lim_{x\to 0+} e^{1/x} = \infty.$

Prob: Find $\lim_{x\to\infty} \dfrac{x^2-4}{x^2+7x+12}$

$\rightarrow \lim_{x\to\infty} \dfrac{x^2-4}{x^2+7x+12}$

$= \lim_{x\to\infty} \dfrac{1-\frac{4}{x^2}}{1+\frac{7}{x}+\frac{12}{x^2}}$

$= \dfrac{1-4\times 0}{1+7\times 0+12\times 0} = 1.$

$$= \frac{1-0}{1+0} = 1$$

$$\therefore \lim_{x\to 0-} \frac{e^{1/x}-1}{e^{1/x}+1} \neq \lim_{x\to 0+} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$\therefore \lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

$$\lim_{x\to 0-} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= \frac{0-1}{0+1} = -1.$$

$$\lim_{x\to 0+} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= \lim_{x\to 0+} \frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+e^{\frac{1}{x}}}$$

$$\lim_{x\to 0+} \frac{x\cdot e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{x\to 0+} \frac{x}{\frac{1}{e^{1/x}}+1}$$

$$= \frac{0}{0+1} = 0.$$

$\therefore \lim_{x\to 0} \frac{xe^{1/x}}{1+e^{1/x}}$ exists & is equal to '0'.

Prob: Evaluate

$$\lim_{x\to 0} \frac{x\cdot e^{1/x}}{1+e^{1/x}}.$$

$\rightarrow$

$$\lim_{x\to 0-} \frac{x\cdot e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{x\cdot e^{1/x}}{1+e^{1/x}} = \frac{0\cdot 0}{1+0} = 0$$

Continuity at a point

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c be a point of A.

f is said to be continuous at $x = c$ iff

$$\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = f(c)$$

or

$$\lim_{x \to c} f(x) = f(c)$$

f is cts or at $x = c$ iff for any $\varepsilon > 0$ $\exists$ some $\delta > 0$ (depending on ε) s.t.

$$|f(x) - f(c)| < \varepsilon$$

whenever

$$|x - c| < \delta$$

i.e. $c - \delta < x < c + \delta$

Continuity from right at $x=c$

$$\lim_{x \to c+} f(x) = f(c)$$

or

$|f(x) - f(c)| < \varepsilon$ whenever $c < x < c+\delta$.

Continuity of $f(x)$ in an open interval $]a,b[$:

Let $p \in]a,b[$ be any pt.

Continuity from left at $x = c$.

A function f is said to be cts from left at $x=c$ if

$$\lim_{x \to c-} f(x) = f(c)$$

or

For any $\varepsilon > 0$, $\exists$ some $\delta > 0$ s.t.

$|f(x) - f(c)| < \varepsilon$ whenever $c - \delta < x < c$

Remark: If a function $y = f(x)$ is not ct (discontinuous) at $x = c$ then it is sufficient to find some $\varepsilon > 0$ and for each $\delta > 0$, we can have some x s.t

$$|f(x) - f(c)| \geq \varepsilon \text{ whenever } |x - c| < \delta$$

Check continuity of f at $x = p$.

Continuity of $y = f(x)$ in a closed interval $[a, b]$:

(I) $\lim_{x \to a+} f(x) = f(a)$

(II) $\lim_{x \to b-} f(x) = f(b)$.

(III) check cont. of $f(x)$ in $]a, b[$

Note: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$, f is said to be discontinuous at $x = c$ iff $\exists$ some seq. $\langle x_n \rangle$ in A converging to 'c' but $\langle f(x_n) \rangle$ does not converge to '$f(c)$'.

Sequential criterion for Continuity:

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$.

f is said to be cts at $x = c$ iff for any seq. $\langle x_n \rangle$ in A converging to 'c', $\langle f(x_n) \rangle \to f(c)$.

Proof: use $\varepsilon-\delta$ definition

to prove that $f(x) = 3x+1$ is ct at $x = 2$.

$\rightarrow$ Let $\varepsilon > 0$ be given.

$$|f(x) - f(2)|$$

$$= |3x+1-7| = |3x-6|$$

$$= 3|x-2|$$

$$|f(x) - f(2)| = 3|x-2| < \varepsilon$$

if $|x-2| < \frac{\varepsilon}{3}$

Choose $\delta = \frac{\varepsilon}{3}$.

$\therefore$ For given $\varepsilon > 0$, $\exists$ some $\delta = \frac{\varepsilon}{3}$ s.t.

$$|f(x) - f(2)| < \varepsilon \text{ when } |x-2| < \delta$$

$\therefore$ f is ct at $x = 2$.

Sol

$$|f(x) - f(2)| = |x - 2|$$

$\therefore$ $|f(x) - f(2)| < \varepsilon$ if $|x-2| < \varepsilon$

Choose $\delta = \varepsilon$, we have

$$|f(x) - f(2)| < \varepsilon \text{ when } |x-2| < \delta$$

$\therefore$ f is cts at $x=2$.

Prob: use ε-δ definition to prove that

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

is cts at $x=2$.

$\rightarrow$ Let $\varepsilon > 0$ be given

$$|f(x) - f(2)| = \left|\frac{x^2-4}{x-2} - 4\right|$$
$$= |x+2-4| = |x-2|$$

$$\lim_{x \to 0} f(x) = 1 \neq f(0)$$

$\therefore$ f is discts at $x=0$,

Removable disc.

(2) Discontinuity of 1st kind or jump disc.

(i) f has a disc. of 1st kind at $x=c$ if $\lim_{x \to c-} f(x)$

Types of Discontinuity

(1) Removable discontinuity

$$\lim_{x \to c} f(x) \neq f(c)$$

Ex:

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

(iii) Disc. of 1st kind from right at $x = 'c'$

$$\lim_{x \to c+} f(x) \text{ exists} \neq f(c)$$

Ex:

$$f(x) = \begin{cases} x^2 - 2, & x > 2 \\ 3 - x, & x < 2 \\ 1, & x = 2 \end{cases}$$

& $\lim_{x \to c+} f(x)$ exists but are not equal.

(ii) Disc. of 1st kind from left at $x = 'c'$

$$\lim_{x \to c-} f(x) \text{ exists} \neq f(c)$$

$$\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$$

Dis. of 1st kind.

$$\lim_{x \to 2^+} f(x) = \lim_{\substack{x \to 2 \\ x > 2}} x^2 - 2$$

$$= 2^2 - 2 = 2$$

$$\lim_{x \to 2^-} f(x) = \lim_{\substack{x \to 2 \\ x < 2}} 3 - x$$

$$= 3 - 2 = 1$$

$$f(2) = 1$$