Maths Optional

By Dhrub Singh Sir

$$p^3 - p = -\frac{dp}{dx}(x + ap^2)$$

$$\Rightarrow p(p^2 - 1) = -\frac{dp}{dx}(x + ap^2)$$

$$\Rightarrow p(1 - p^2) = \frac{dp}{dx}(x + ap^2)$$

$$\Rightarrow \frac{dx}{dp} p(1 - p^2) = x + ap^2$$

$$\Rightarrow \frac{dx}{dp} = \frac{x}{p(1-p^2)} + \frac{a p^{\cancel{2}}}{\cancel{p}(1-p^2)}$$

Prob:

$$x - yp = ap^2$$

$$\Rightarrow yp = x - ap^2$$

$$\Rightarrow y = \frac{x}{p} - ap \quad \text{--- (1)}$$

Diff. w.r.t. 'x'.

$$p \frac{dy}{dx} = \frac{p - x \cdot \frac{dp}{dx}}{p^2} - a\frac{dp}{dx}$$

$$p^3 = p - x\frac{dp}{dx} - ap^2\frac{dp}{dx}$$

$$\frac{1}{\cancel{p(1-p)(1+p)}} = \frac{A(1-p)(1+p) + B\cdot p\cdot(1+p) + C\cdot p\cdot(1-p)}{\cancel{p(1-p)(1+p)}}$$

$$1 = A(1-p)(1+p) + B\,p\,(1+p) + C\cdot p\cdot(1-p)$$

$\underline{p=0}$ $\quad 1 = A$

$\underline{p=1}$ $\quad 1 = 2B \Rightarrow B = \frac{1}{2}$

$\underline{p=-1}$ $\quad 1 = -2C \Rightarrow C = -\frac{1}{2}$

$$\frac{dx}{dp} - \frac{x}{p(1-p^2)} = \frac{ap}{1-p^2}$$

Linear in x

$$-\int \frac{1}{p(1-p)(1+p)}\,dp$$

$$\frac{1}{p(1-p)(1+p)} = \frac{A}{p} + \frac{B}{1-p} + \frac{C}{1+p}$$

$$\text{IF} = e^{\log \frac{\sqrt{1-p^2}}{p}} = \frac{\sqrt{1-p^2}}{p}$$

$$x \cdot \frac{\sqrt{1-p^2}}{p} = \int \frac{\sqrt{1-p^2}}{\cancel{p}} \times \frac{a\cancel{p}}{1-p^2}\,dp \quad + \text{const}$$

$$= a\int \frac{dp}{\sqrt{1-p^2}} + \text{const}$$

$$x \cdot \frac{\sqrt{1-p^2}}{p} = a\sin^{-1}p + c$$

$$x = \frac{a\,p\,\sin^{-1}p}{\sqrt{1-p^2}} + \frac{cp}{\sqrt{1-p^2}}$$

$$= \frac{p}{\sqrt{1-p^2}}\left[a\sin^{-1}p + c\right]$$

$$-\int \frac{dp}{p(1-p)(1+p)}$$

$$= -\int \left[\frac{1}{p} + \frac{1}{2}\cdot\frac{1}{1-p} - \frac{1}{2}\cdot\frac{1}{1+p}\right] dp$$

$$= -\left[\log p - \frac{1}{2}\log(1-p) - \frac{1}{2}\log(1+p)\right]$$

$$= -\left[\log p - \log\sqrt{1-p^2}\right]$$

$$= \log\frac{\sqrt{1-p^2}}{p}.$$

Putting in (1).

$$x = \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right)$$

$$y = \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right) - ap.$$

Int;

$$x = \int \frac{dp}{p(p-3)} + \text{const.} \quad \text{(ii)}$$

$$\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3}$$

$$1 = A(p-3) + Bp$$

$\underline{p=0}$ $\quad A = -\frac{1}{3}$

$\underline{p=3}$ $\quad B = \frac{1}{3}$

Prob: Solve

$$y = 3x + \log p \quad \text{--- (i)}$$

$\longrightarrow$ Diff. w.r.t x

$$\frac{dy}{dx} = 3 + \frac{1}{p} \cdot \frac{dp}{dx}$$

$$\Rightarrow p - 3 = \frac{1}{p} \frac{dp}{dx}$$

$$\Rightarrow dx = \frac{dp}{p(p-3)}$$

$$\Rightarrow \frac{p-3}{p} = e^{3x} \cdot c$$

$$\Rightarrow 1 - \frac{3}{p} = c \cdot e^{3x}$$

$$\Rightarrow 1 - ce^{3x} = \frac{3}{p}$$

$$\Rightarrow \frac{p}{3} = \frac{1}{1 - c \cdot e^{3x}}$$

$$\Rightarrow p = \frac{3}{1 - ce^{3x}}$$

using this in (1),

$$y = 3x + \log\left(\frac{3}{1 - ce^{3x}}\right)$$

$$\int \frac{dp}{p(p-3)} = \frac{1}{3}\left[\int \frac{dp}{p-3} - \int \frac{dp}{p}\right]$$

$$= \frac{1}{3}\left[\log(p-3) - \log p\right]$$

$$= \frac{1}{3} \log \frac{p-3}{p}$$

(11) $\Rightarrow$

$$x = \frac{1}{3} \log \frac{p-3}{p} + \text{const}$$

$$3x + \text{const} = \log \frac{p-3}{p}$$

$$\frac{p-3}{p} = e^{3x + \text{const}}$$

$$2x(1-p)\frac{dp}{dx}+p(1-p)=0$$

$$\Rightarrow \underline{(1-p)}\left[2x\frac{dp}{dx}+p\right]=0$$

$$\Rightarrow 2x\frac{dp}{dx}+p=0$$

$$\Rightarrow 2x\,dp+p\,dx=0$$

$$\Rightarrow \frac{dp}{p}+\frac{dx}{2x}=0$$

Int

$$\log p+\frac{1}{2}\log x=\text{const}$$

<u>Prob:</u> Solve $y=2px-x\cdot p^2$ — (1)

$\rightarrow$ Diff. w.r.t. 'x',

$$p=2p+2x\frac{dp}{dx}-p^2-2px\cdot\frac{dp}{dx}$$

$$\Rightarrow -p+p^2=2x(1-p)\frac{dp}{dx}$$

$$\Rightarrow -p(1-p)=2x(1-p)\frac{dp}{dx}$$

Prob: solve

$$x p^2 - 2yp + ax = 0 \quad \text{—} (1)$$

$$\Rightarrow \quad 2yp = xp^2 + ax$$

$$\Rightarrow y = \frac{xp^{\cancel{2}}}{2\cancel{p}} + \frac{ax}{2p}$$

Diff. w.r.t. x,

$$p = \frac{1}{2}\left[p + x\frac{dp}{dx}\right] + \frac{a}{2}\left[\frac{p - x\frac{dp}{dx}}{p^2}\right]$$

$$\log p + \log\sqrt{x} = \text{const}$$

$$\Rightarrow \log p\cdot\sqrt{x} = \text{const}$$

$$\Rightarrow p\cdot\sqrt{x} = c$$

$$\Rightarrow p = \frac{c}{\sqrt{x}} \quad \text{———} (11)$$

using this in (1)

$$y = 2\times\frac{c}{\sqrt{x}}\times x - x\cdot\left(\frac{c}{\sqrt{x}}\right)^2$$

$$y = 2c\sqrt{x} - c^2$$

$$p\,dx - x\,dp = 0$$

$$\Rightarrow \frac{dx}{x} - \frac{dp}{p} = 0$$

Int

$$\log x - \log p = \text{const}$$

$$\Rightarrow \log\frac{x}{p} = \text{const}$$

$$\Rightarrow \frac{x}{p} = \text{Const}$$

$$\Rightarrow \frac{p}{x} = c \Rightarrow \underline{p = cx}$$

$$\frac{1}{\cancel{2}}p = \frac{1}{\cancel{2}}x\frac{dp}{dx} + \frac{a}{\cancel{2}}\left[\frac{p - x\frac{dp}{dx}}{p^2}\right]$$

$$\Rightarrow p - x\frac{dp}{dx} = \frac{a}{p^2}\left[p - x\frac{dp}{dx}\right]$$

$$\Rightarrow \left[p - x\frac{dp}{dx}\right]\underbrace{\left[1 - \frac{a}{p^2}\right]}_{\text{sing. soln.}} = 0$$

$$\Rightarrow p - x\frac{dp}{dx} = 0$$

Solvable for x

Prob: Solve $y = 2px + y^2p^3$ — (1)

$\Rightarrow 2px = y - y^2p^3$

$\Rightarrow x = \frac{y}{2p} - \frac{y^2p^3}{2p}$

i.e. $x = \frac{y}{2p} - \frac{1}{2}y^2p^2$

Diff. w.r.t y.

Using this in (1),

$x \cdot (cx)^2 - 2y \times cx + ax = 0$

$\Rightarrow c^2x^3 - 2cxy + ax = 0$

$\Rightarrow c^2x^2 - 2cy + a = 0$

$\Rightarrow 2cy = c^2x^2 + a$

$\Rightarrow \boxed{2y = cx^2 + \frac{a}{c}}$

$$\underbrace{\left[\frac{1}{p}+2yp^2\right]}_{\text{sing. sol}^n}\left[1+\frac{y}{p}\frac{dp}{dy}\right]=0$$

$$\Rightarrow 1+\frac{y}{p}\frac{dp}{dy}=0$$

$$\Rightarrow dy+\frac{y}{p}dp=0$$

$$\Rightarrow \frac{dy}{y}+\frac{dp}{p}=0$$

Int

$$\log y+\log p=\text{const}$$

$$\log py=\text{const}$$

$$py=c$$

$$\frac{1}{p}=\frac{1}{2}\left[\frac{p-y\frac{dp}{dy}}{p^2}-\left(2yp^2+y^2\cdot 2p\cdot\frac{dp}{dy}\right)\right]$$

$$\Rightarrow \frac{2}{p}=\frac{p-y\frac{dp}{dy}}{p^2}-2yp\left[p+y\frac{dp}{dy}\right]$$

$$\Rightarrow \frac{1}{p}=-\frac{y}{p^2}\frac{dp}{dy}-2yp^2-2y^2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p}+2yp^2=-\frac{dp}{dy}\left[\frac{y}{p^2}+2y^2p\right]$$

$$\Rightarrow \frac{1}{p}+2yp^2=-\frac{y}{p}\frac{dp}{dy}\left[\frac{1}{p}+2yp^2\right]$$

Prob: Solve $y = 2px + p^2 y$

HW

Ans: $y^2 = 2cx + c^2$

HW

Prob

$y^2 + xyp - x^2p^2 = 0$

$\Rightarrow x^2p^2 - xyp - y^2 = 0$

$p = \dfrac{xy \pm \sqrt{x^2y^2 + 4x^2y^2}}{2x^2}$

$p = \dfrac{c}{y}$ — (11)

using (11) in (1)

$y = 2 \times \dfrac{c}{y} \times x + y^2 \times \dfrac{c^3}{y^3}$

$\Rightarrow y = \dfrac{2cx}{y} + \dfrac{c^3}{y}$

$\Rightarrow y^2 = 2cx + c^3$

Int.

$$\log y = \frac{1}{2}(1+\sqrt{5})\log x + \text{cnst}$$

$$\Rightarrow \log y = \log x^{\frac{1}{2}(1+\sqrt{5})} + \text{cnst}$$

$$\Rightarrow \log \frac{y}{x^{\frac{1}{2}(1+\sqrt{5})}} = \text{cnst}$$

$$\Rightarrow \frac{y}{x^{\frac{1}{2}(1+\sqrt{5})}} = c$$

$$\Rightarrow y = c \cdot x^{\frac{1}{2}(1+\sqrt{5})}$$

$$p = \frac{xy + \sqrt{5}\,xy}{2x^2}$$

$$\Rightarrow p = \frac{1}{2}\left[\frac{y}{x} + \sqrt{5}\,\frac{y}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{(1+\sqrt{5})y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{1}{2}(1+\sqrt{5}) \cdot \frac{dx}{x}$$

HW

Prob: $x = y + p^2$

Ans: $x = c - \left[2p + 2\log(p-1)\right]$

$y = c - \left[p^2 + 2p + 2\log(p-1)\right]$

Similarly,

$$p = \frac{xy(1-\sqrt{5})}{2x^2}$$

$$y = c \cdot x^{\frac{1}{2}(1-\sqrt{5})}$$

Soln

$$\left[y - c\,x^{\frac{1}{2}(1+5)}\right]\left[y - c\,x^{\frac{1}{2}(1-\sqrt{5})}\right]$$

$= 0$