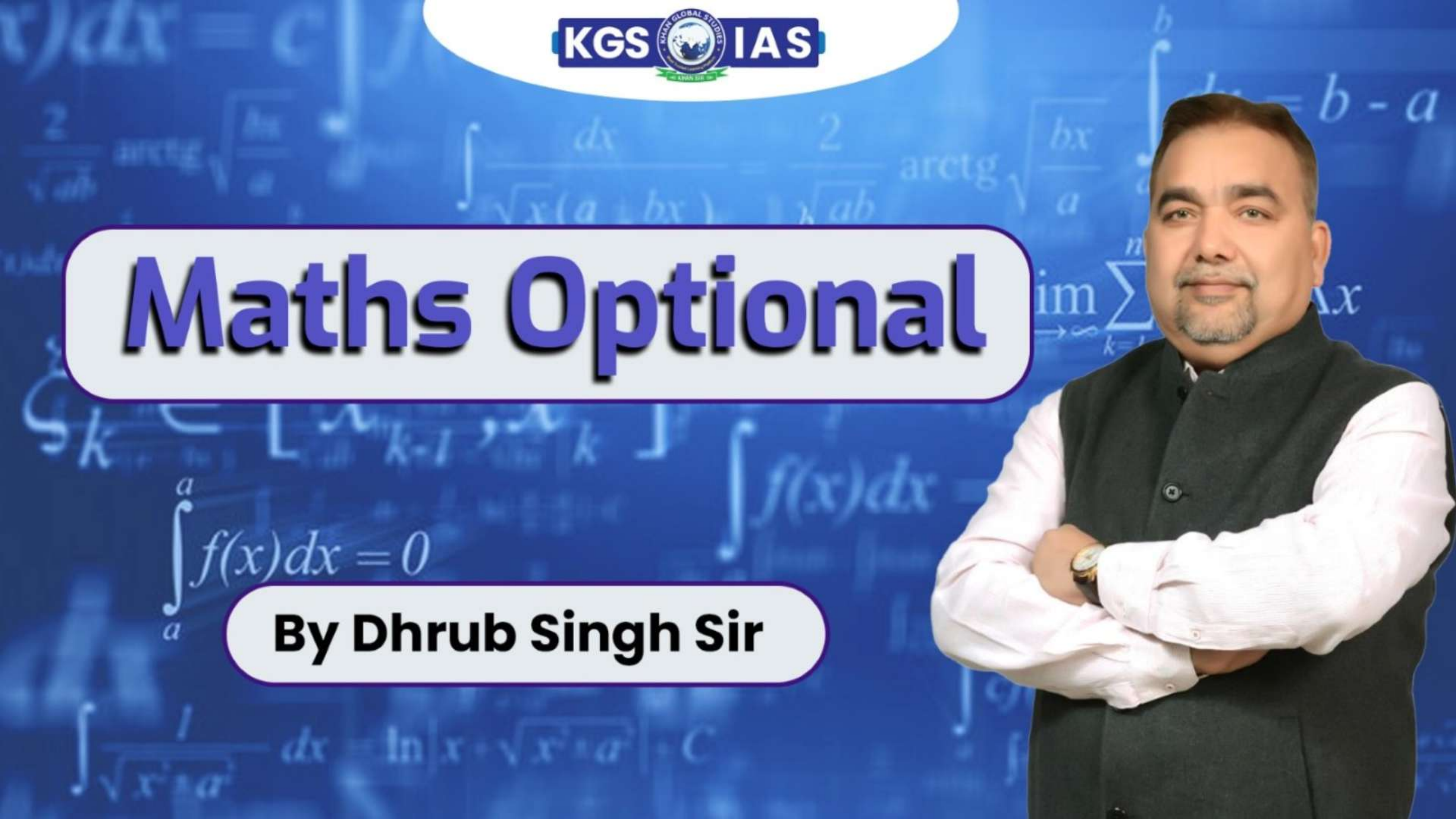

KGS IAS
Maths Optional
By Dhrub Singh Sir

Conditions:

(i) t is a lower bound of S.
i.e. $t \le x \ \forall x \in S$

(ii) If w is a lower bound of S then
$w \le t$.

Notation: $\inf S$.

Greatest lower bound (g.l.b.) or infimum of a set:

Let S be a non-empty subset of $\mathbb{R}$. If S is bounded below then a number t is said to be the greatest lower bound (g.l.b) or infimum of the set S if it satisfies the following two i.e., $\exists$ an element $x \in S$

s.t. $x < t + \varepsilon$.

Least upper bound (l.u.b) or supremum of a set

Let S be a non-empty subset of $\mathbb{R}$. And S is bounded above. A number t is said to be least upper bound (l.u.b) or

Ex: (i) $\mathbb{N}$:

$\inf \mathbb{N} = 1$

(ii) $S =]0, 1[$

$\inf S = 0$

Note: If $\inf S = t$ then for each $\varepsilon > 0$ (may be very small), the number $t + \varepsilon$ is not a lower bound of S

Notation: sup S

Ex: ① $S =]0, 1[$

$x < 1 \ \forall \ x \in S$

$\sup S = 1$

supremum of the set S iff it satisfies the following conditions:

(i) t is an upper bound of S.

i.e. $x \leq t \ \forall x \in S$

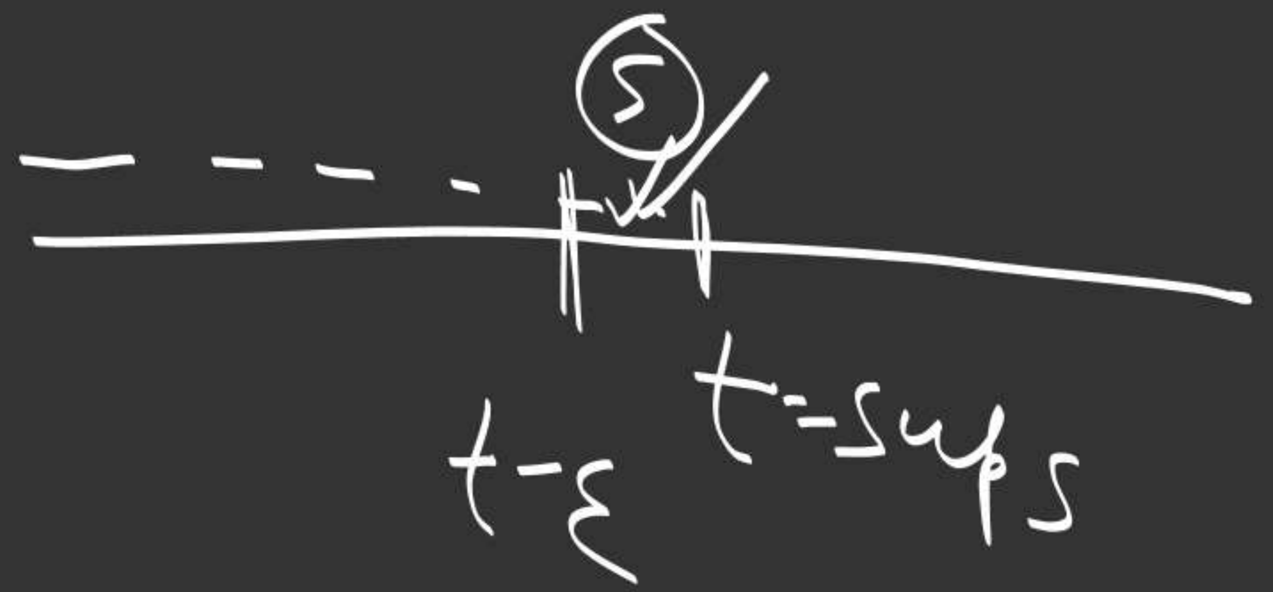
(ii) If t' is an upper bound of S then $t \leq t'$.

a number then $t-\varepsilon$ cannot be an upper bound of S

i.e. $\exists$ an element $x \in S$

s.t.

$$t-\varepsilon < x$$



(11) $\mathbb{N}$

$\sup \mathbb{N}$ doesn't exist.

Note: If $\sup S = t$ and $\varepsilon > 0$ (may be very small) be

$0 < x \leq 1 \quad \forall x \in S$

$\inf S = 0 \notin S$

$\sup S = 1 \in S$

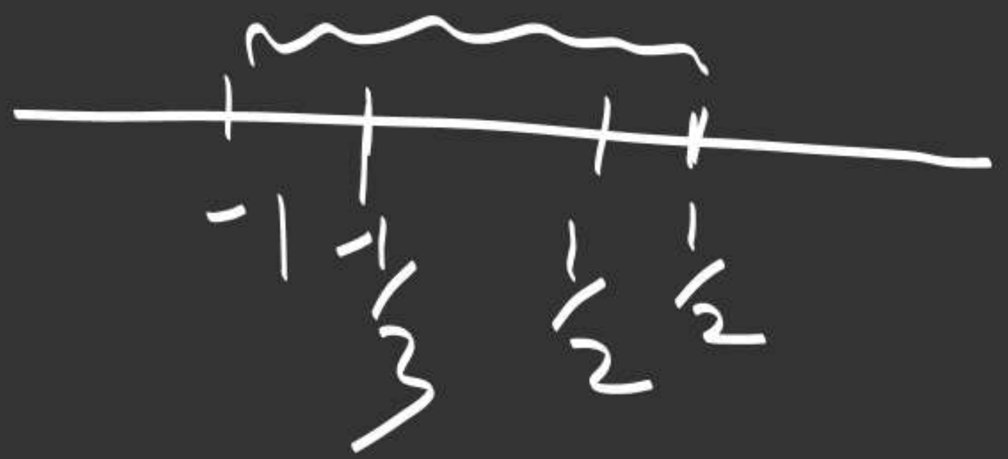
(11) $S = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$

$= \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, \ldots \right\}$

prob: Find the infimum and supremum of the following sets.

(1) $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}$



$$-1 \le x \le \frac{1}{2} \quad \forall x \in S$$

$$\inf S = -1, \ \sup S = \frac{1}{2}$$

(IV) $S = \left\{ \frac{n+1}{n} : n \in \mathbb{N} \right\}$

$$= \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}$$

$$-1 \le x < 0 \quad \forall x \in S.$$

$$\therefore \inf S = -1$$

$$\sup S = 0$$

(III) $S = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

$$= \left\{ -1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \dots \right\}$$

(v) $S = \{3\}$

$\sup S = 3$

$\inf S = 3$

Greatest member of a set

A number G is called the greatest member of a non-empty set $S \subseteq \mathbb{R}$ if the following conditions hold:

$= \{2, 1+\frac{1}{2}, 1+\frac{1}{3}, 1+\frac{1}{4}, \ldots\}$

$1 < x \le 2, \ \forall x \in S$

$\inf S = 1$

$\sup S = 2$

$S \subseteq \mathbb{R}$ if the following conditions hold:

(i) $g \leq x \quad \forall x \in S$

(ii) $g \in S$

<u>Ex</u>: $S = \{1, 2, --10\}$.

Smallest member $= 1$.

(i) $x \leq g, \quad \forall x \in S$

(ii) $g \in S$

<u>Ex</u>: $S = \{1, 2, — 10\}$

Greatest member $= 10$

<u>Smallest member of a set</u>

A number g is called the smallest member of a non-empty set

Complete ordered field: The set $\mathbb{R}$ of real numbers is a complete ordered field. Because $\mathbb{R}$ satisfies

(I) Field axioms.

(II) Order axioms.

(III) Completeness axioms.

Completeness property of $\mathbb{R}$

→ Every non-empty subset of $\mathbb{R}$ which is bounded above has the supremum in $\mathbb{R}$.

→ Every non-empty subset of $\mathbb{R}$ which is bounded below has the infimum in $\mathbb{R}$.

real numbers and if $a > 0$ then $\exists$ a +ve integer n s.t.

$$na > b.$$

<u>Proof</u>: Let a and b are real numbers.

Let $a > 0$

Let if ~~possible~~,

$$na \leq b \quad \forall n \in \mathbb{Z}^+$$

<u>Note</u>: The set of rational numbers $\mathbb{Q}$ is an ordered field but not complete.

<u>Ex</u>: $S = \{x \in \mathbb{Q}^+ : 0 < x^2 < 2\}$

<u>The Archimedian property</u>

<u>Stat</u>: If a & b be any two

$$na \le M \quad \forall n \in \mathbb{Z}^+$$

$$\Rightarrow (n+1)a \le M \quad \forall n \in \mathbb{Z}^+$$

$$\Rightarrow na + a \le M \quad ''$$

$$\Rightarrow na \le M - a \quad ''$$

As $a > 0$

$\therefore M - a < M$

thus $M-a$ is an upper bound of S. which is a contradiction

Consider the set

$$S = \{na : n \in \mathbb{Z}^+\}.$$

$$x \le b, \quad \forall x \in S$$

$\therefore$ S is bounded above.

$\therefore$ By completeness Prop. of real numbers.

$\exists$ a real number M s.t.

n s.t. $n > b$.

Proof: Take $a = 1$

By the main th.,

$$n \cdot 1 > b$$

i.e. $n > b$.

to the fact that

$\sup S = M$.

$\therefore$ our supposition is wrong.

And so $\exists$ a +ve int. n s.t.

$$na > b.$$

Cor: Let b be any real number then $\exists$ a +ve int.

(iii) $|x| = |-x|$

(iv) $|x| = \sqrt{x^2}$

Result: If x and y are any two real numbers then

(i) $|x+y| \leq |x| + |y|$

(ii) $|x-y| \geq \big||x| - |y|\big|$

Absolute value (Modulus) of a real number:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties: (i) $|x| = \max\{x, -x\}$

(ii) $|x|^2 = x^2$

Proof:

(I) $|x+y|^2 = (x+y)^2$

$= x^2 + y^2 + 2xy$

$= |x|^2 + |y|^2 + 2xy$

$\leq |x|^2 + |y|^2 + 2|x|\cdot|y|$

$= (|x| + |y|)^2$

i.e. $|x+y|^2 \leq (|x|+|y|)^2$

Taking +ve square roots of both sides

$|x+y| \leq |x| + |y|$

(III) $|xy| = |x|\cdot|y|$

(IV) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \quad y \neq 0$

(V) $|x-y| \leq |x| + |y|$

(VI) $|x| < \varepsilon \iff -\varepsilon < x < \varepsilon$

(VII) $|x-a| < \varepsilon \iff$

$a - \varepsilon < x < a + \varepsilon$

i.e. $|x-y| \geq -(|x|-|y|)$ —— (ii)

By (i) & (ii),

$$|x-y| \geq \max\{|x|-|y|, -(|x|-|y|)\}$$

i.e. $|x-y| \geq \big||x|-|y|\big|$

(i) $|x| = |(x-y)+y| \leq |x-y|+|y|$

i.e. $|x| \leq |x-y|+|y|$

i.e. $|x|-|y| \leq |x-y|$

i.e. $|x-y| \geq |x|-|y|$ —— (i)

Changing the role of x & y in (i),

$$|y-x| \geq |y|-|x|$$

(VII) $|x-a| < r$

$\iff -r < x-a < r$

$\iff a-r < x < a+r$

(VI) $|x| < r$

$\iff \max\{x, -x\} < r$

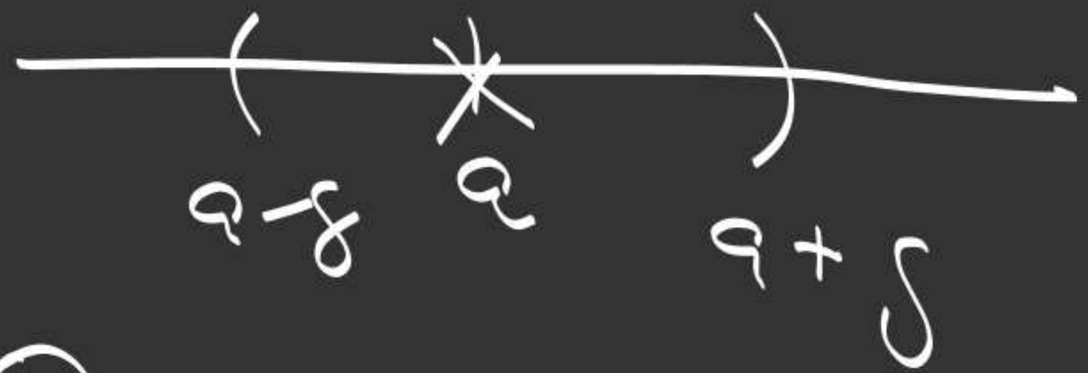
$\iff x < r, \; -x < r$

$\iff x < r, \quad x > -r$

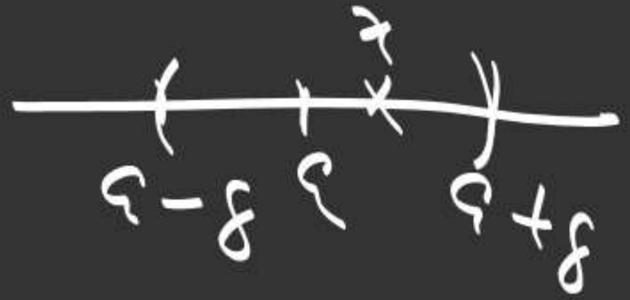
$\iff -r < x < r$

Note: (i) Deleted nbhd of a point 'a' is

$$N_\delta(a) \sim \{a\}$$



(ii) $x \in N_\delta(a)$



Neighbourhood of a point:

If a is any real number and $\delta > 0$ (May be very small), then the open interval $(a-\delta, a+\delta)$ is called a nbhd of the point 'a'.

Notation: $N_\delta(a)$ or $N(\delta, a)$

$a-\delta < x < a+\delta$

i.e. $|x-a| < \delta$

(iii) $x \in N_\delta(a) \sim \{a\}$.

$\iff$ $a-\delta < x < a+\delta$, $x \neq a$

$\iff$ $|x-a| < \delta$, $x \neq a$

$\iff$ $0 < |x-a| < \delta$