Chapter 03 Trigonometric Ratios

CONTENT

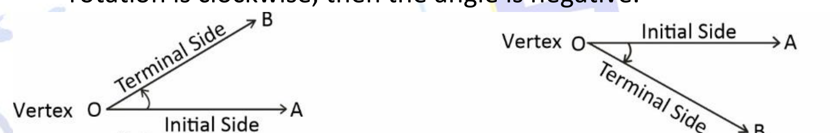
Trigonometric Ratios is also known as compound angle or trigonometry Phase-I.

1. INTRODUCTION

The word 'trigonometry' is derived from the Greek words 'trigon' and ' metron' and it means 'measuring the sides and angles of a triangle'.

Angle :

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative.



(i) Positive angle (anticlockwise measurement)

(ii) Negative angle (clockwise measurement)

Systems For Measurement of Angles :

An angle can be measured in the following systems.

(1) Sexagesimal System (British System) : In this system $\frac{1}{360}$ of a complete circular turn is called a degree(°), $\frac{1}{60}$ of a degree is called a minute (′) and $\frac{1}{60}$ of a minute is called a second (″).

One right angle = 90°, 1° = 60′, 1′ = 60″

(2) Centesimal System (French System) : In this system $\frac{1}{400}$ of a complete circular turn is called a grade(^{g}), $\frac{1}{100}$ of a grade is called a minute (‘) and $\frac{1}{100}$ of a minute is called a second (“).

$\therefore$ One right angle = 100^g; 1^g = 100‘; 1‘ = 100“

DETECTIVE MIND

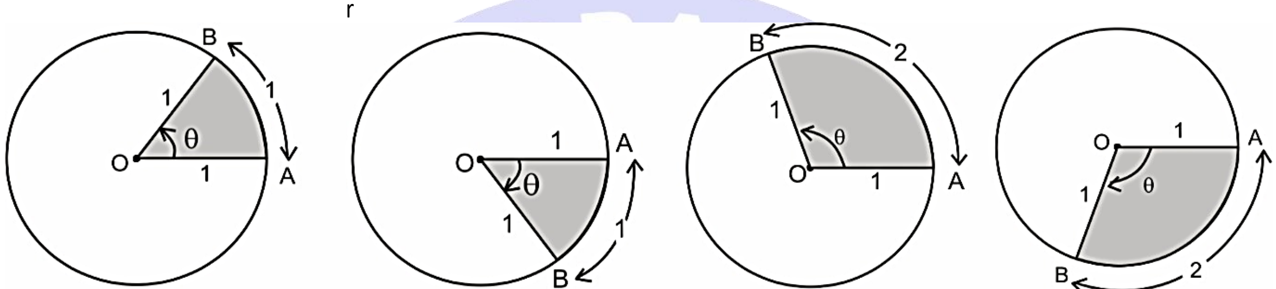
The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.

(3) Circular System (Radian Measurement)

The angle subtended by an arc of a circle whose length is equal to the radius of the circle at the centre of the circle is called a radian. In this system the unit of measurement is radian (^{c}).

As the circumference of a circle of radius 1 unit is 2π, therefore one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length ℓ will subtend an angle whose measure is $\frac{\ell}{r}$ radian. Thus, if in a circle of radius r, arc of length ℓ subtends an angle θ radian at the centre, we have $\theta = \frac{\ell}{r}$ or $\ell = r\theta$.



$\theta = 1$ radian (i) $\qquad$ $\theta = -1$ radian (ii) $\qquad$ $\theta = 2$ radian (iii) $\qquad$ $\theta = -2$ radian (iv)

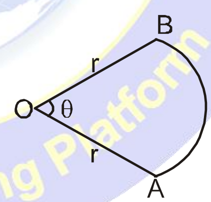
DETECTIVE MIND

If no symbol is mentioned while showing measurement of angle, then it is considered to be measured in radians.

e.g. $\theta = 15$ implies 15 radian

Area of circular sector :

Area $= \frac{1}{2} r^2\theta$ sq. units



Relation between radian, degree and grade :

$\frac{\pi}{2}$ radian $= 90° = 100^g$

Trigonometric Ratios for Acute Angles :

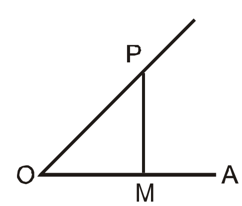
Let a revolving ray OP starts from OA and revolves into the position OP, thus tracing out the angle AOP.
In the revolving ray take any point P and draw PM perpendicular to the initial ray OA.
In the right angle triangle MOP, OP is the hypotenuse, PM is the perpendicular, and OM is the base.
The trigonometrical ratios, or functions, of the angle AOP are defined as follows :

$\frac{MP}{OP}$, i.e., $\frac{\text{Perp.}}{\text{Hyp.}}$, is called the Sine of the angle AOP;

$\frac{OM}{OP}$, i.e. $\frac{\text{Base}}{\text{Hyp.}}$, is called the Cosine of the angle AOP;

$\frac{MP}{OM}$, i.e. $\frac{\text{Perp.}}{\text{Base}}$, is called the Tangent of the angle AOP;

$\frac{OM}{MP}$, i.e. $\frac{\text{Base}}{\text{Perp.}}$, is called the Cotangent of the angle AOP;



$\frac{OP}{OM}$, i.e. $\frac{\text{Hyp.}}{\text{Base}}$, is called the Secant of the angle AOP;

$\frac{OP}{MP}$, i.e. $\frac{\text{Hyp.}}{\text{Perp.}}$, is called the Cosecant of the angle AOP;

2. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1 \quad \forall \quad \theta \in R$

(2) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1 \quad \forall \quad \theta \in R$

(3) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$; $|\operatorname{cosec}\theta| \ge 1 \quad \forall \quad \theta \in R$

3. IMPORTANT TRIGONOMETRIC RATIOS :

(1) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(2) $\sin \frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos \frac{(2n+1)\pi}{2} = 0$ where $n \in I$

(3) $\sin 15°$ or $\sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos \frac{5\pi}{12}$;

$\cos 15°$ or $\cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin \frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(4) $\sin \frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos \frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\tan \frac{\pi}{8} = \sqrt{2}-1$; $\tan \frac{3\pi}{8} = \sqrt{2}+1$

(5) $\sin \frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$ & $\cos 36°$ or $\cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54°$

4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES**.

(1) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(2) $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$

(3) $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$

(4) $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$

(5) $\sin(180°+\theta) = -\sin\theta$; $\cos(180°+\theta) = -\cos\theta$

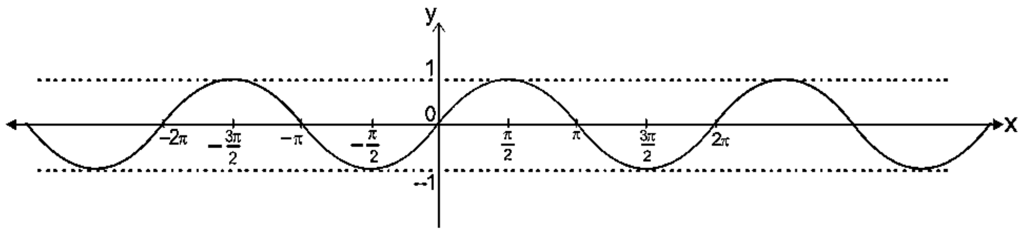
(6) $\sin(270°-\theta) = -\cos\theta$; $\cos(270°-\theta) = -\sin\theta$

(7) $\sin(270°+\theta) = -\cos\theta$; $\cos(270°+\theta) = \sin\theta$

5. TRIGONOMETRIC FUNCTIONS:

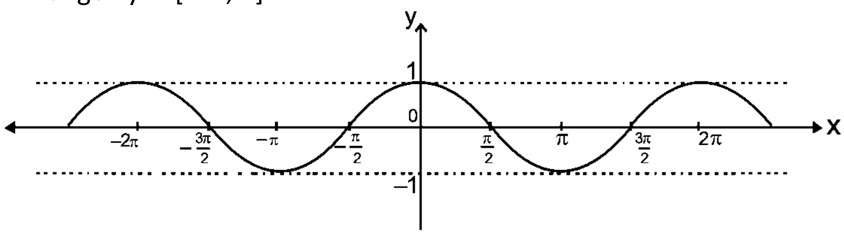
(i) **y = sin x** Domain : $x \in R$

Range : $y \in [-1, 1]$



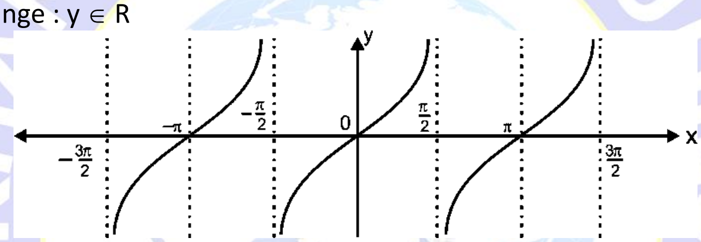
(ii) y = cos x Domain : $x \in R$

Range : $y \in [-1, 1]$



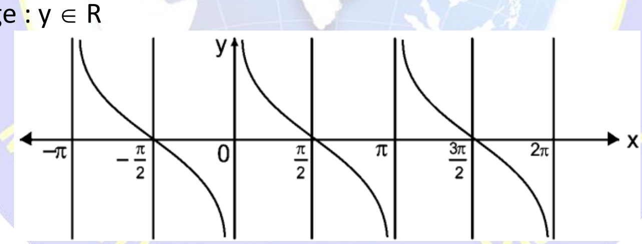
(iii) y = tan x Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$

Range : $y \in R$



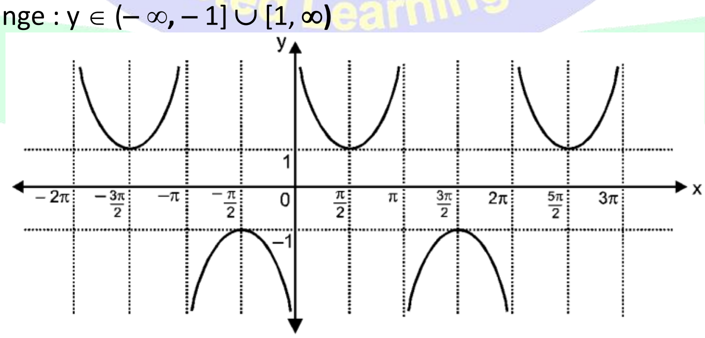
(iv) y = cot x Domain : $x \in R - \{n\pi\}$, $n \in I$

Range : $y \in R$



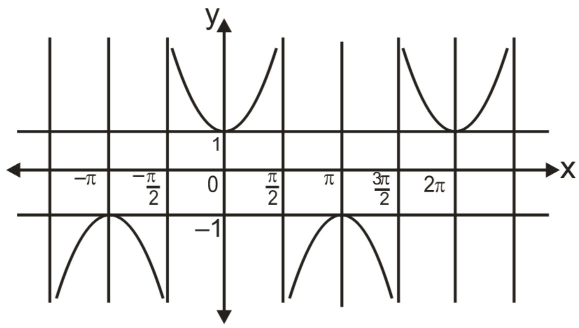
(v) y = cosec x Domain : $x \in R - \{n\pi\}$, $n \in I$

Range : $y \in (-\infty, -1] \cup [1, \infty)$



(vi) y = sec x Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$

Range : $y \in (-\infty, -1] \cup [1, \infty)$



6. TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES :

(1) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(2) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(3) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \,.\, \sin(A-B)$

(4) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \,.\, \cos(A-B)$

(5) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(6) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

7. TRANSFORMATION FORMULAE :

(i) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

(ii) $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$

(iii) $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$

(iv) $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$

(v) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(vi) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

(vii) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(viii) $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$

SOLVED EXAMPLES

Example : 1 The value of the expression -

$$1 - \frac{\sin^2 y}{1+\cos y} + \frac{1+\cos y}{\sin y} - \frac{\sin y}{1-\cos y} \text{ is equal to -}$$

(1) 0 (2) 1 (3) sin y (4) cos y

Solution :
$$1 - \frac{\sin^2 y}{1+\cos y} + \frac{1+\cos y}{\sin y} - \frac{\sin y}{1-\cos y} = \frac{1+\cos y - \sin^2 y}{1+\cos y} + \frac{1-\cos^2 y - \sin^2 y}{\sin y\,(1-\cos y)} = \frac{\cos y + \cos^2 y}{1+\cos y} + 0 = \cos y$$

Example : 2 If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, then $(m^2 n)^{2/3} + (n^2 m)^{2/3}$ equals to -

(1) 0 (2) 1 (3) –1 (4) 2

Solution : $\operatorname{cosec}\theta - \sin\theta = m$

$$m = \frac{1}{\sin\theta} - \sin\theta = \frac{\cos^2\theta}{\sin\theta} \quad \text{...(i)}$$

$$n = \frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta} \quad \text{...(ii)}$$

from (i) and (ii)

$m \times n = \frac{\cos^2\theta}{\sin\theta} \cdot \frac{\sin^2\theta}{\cos\theta} = \sin\theta\cos\theta$

from (i) $\cos^2\theta = m \cdot \sin\theta$

or $\cos^3\theta = m \sin\theta \cos\theta = m \cdot (mn) = m^2n$

Similarly, $\sin^3\theta = n^2m$

Since, $\sin^2\theta + \cos^2\theta = 1$

$(n^2m)^{2/3} + (m^2n)^{2/3} = 1$

Example : 3 Prove that

(i) $\sin(45° + A)\cos(45° - B) + \cos(45° + A)\sin(45° - B) = \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{3\pi}{4}+\theta\right) = -1$

Solution : (i) Clearly $\sin(45° + A)\cos(45° - B) + \cos(45° + A)\sin(45° - B)$

$= \sin(45° + A + 45° - B) = \sin(90° + A - B) = \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right) \times \tan\left(\frac{3\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta} \times \frac{-1+\tan\theta}{1+\tan\theta} = -1$

Example : 4 Prove that $\cos 7A + \cos 8A = 2\cos\left(\frac{15A}{2}\right)\cos\left(\frac{A}{2}\right)$

Solution : L.H.S. $\cos 7A + \cos 8A = 2\cos\left(\frac{15A}{2}\right)\cos\left(\frac{A}{2}\right)$

$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

Example : 5 Find the value of $2\sin 3\theta \sin\theta - \cos 2\theta + \cos 4\theta$

Solution : $2\sin 3\theta \sin\theta - \cos 2\theta + \cos 4\theta = 2\sin 3\theta \sin\theta - 2\sin 3\theta \sin\theta = 0$

Example : 6 Prove that

(i) $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

(ii) If $A + B = 45°$ then prove that $(1 + \tan A)(1 + \tan B) = 2$

Solution : (i) $\frac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta} = \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta} = \frac{2\sin 2\theta\cos 5\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta$

(ii) $A + B = 45°$

$\tan(A + B) = 1 \quad \Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = 1$

$\tan A + \tan B = 1 - \tan A\tan B \quad \Rightarrow \quad \tan A + \tan B + \tan A\tan B + 1 = 2$

$(1 + \tan A)(1 + \tan B) = 2$

Example : 7 $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to

(1) 0 (2) 1 (3) –2 (4) None of these

Solution : $2[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 3[(\sin^2\theta + \cos^2\theta)^2] - 2\sin^2\theta\cos^2\theta + 1$

$\Rightarrow 2[1 - 3\sin^2\theta\cos^2\theta] - 3[1 - 2\sin^2\theta\cos^2\theta] + 1 \Rightarrow 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 0$

Example : 8 If $3\sin\theta + 4\cos\theta = 5$ then the value of $4\sin\theta - 3\cos\theta$ is

(1) 0 (2) 1 (3) 2 (4) 3

Solution : Let $4 \sin \theta - 3 \cos \theta = a$

Thus we want to eliminate θ from both $3 \sin\theta + 4 \cos \theta = 5$ and $4 \sin \theta - 3 \cos \theta = a$, i.e. squaring and adding these equations. We get $(3 \sin \theta + 4 \cos \theta)^2 + (4 \sin \theta - 3 \cos \theta)^2 = 25 + a^2$

$9 \sin^2 \theta + 16 \cos^2 \theta + 24 \sin \theta \cos \theta + 16 \sin^2 \theta + 9 \cos^2 \theta - 24 \cos \theta \sin \theta = 25 + a^2$

$9 + 16 = 25 + a^2$

or $a^2 = 0$

$a = 0$

$4 \sin \theta - 3 \cos \theta = 0$

Example : 9 If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$. Then the value of $x^2 + y^2 + z^2$ is equal to

(1) $2r^2$ (2) r^2 (3) 0 (4) none of these

Solution : Here

$x^2 + y^2 + z^2 = r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2\theta . \sin^2 \phi + r^2 \cos^2\theta$

$= r^2 \sin^2\theta (\cos^2 \phi + \sin^2\phi) + r^2 \cos^2 \theta = r^2 \sin^2\theta + r^2 \cos^2 \theta = r^2 (\sin^2\theta + \cos^2\theta) = r^2$

$x^2 + y^2 + z^2 = r^2$

Example : 10 $\sqrt{3}$ cosec20° – sec 20° is equal to

(1) 0 (2) 1 (3) 2 (4) 4

Solution : $\frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ . \cos 20^\circ} = \frac{4\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{2\sin 20^\circ \cos 20^\circ}$

$= 4.\frac{(\sin 60^\circ.\cos 20^\circ - \cos 60^\circ.\sin 20^\circ)}{\sin 40^\circ} = 4\ \frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ} = 4\ .\frac{\sin 40^\circ}{\sin 40^\circ} = 4$

Example : 11 $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$ is equal to

(1) 1/2 (2) –1/2 (3) 2 (4) –2

Solution : The expression

$= (\sin 78^\circ - \sin 42^\circ) - (\sin 66^\circ - \sin 6^\circ) = 2\cos (60^\circ) \sin (18^\circ) - 2 \cos 36^\circ . \sin 30^\circ$

$= \sin 18^\circ - \cos 36^\circ = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$

Example : 12 Find set of all possible values of α in $[-\pi , \pi]$ such that $\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}$ is equal to $(\sec \alpha - \tan \alpha)$.

(1) $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$ (2) $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ (3) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (4) None of these

Solution : Clearly. $\alpha \neq \pm\pi/2$

as $\sec \alpha - \tan \alpha = \frac{1-\sin\alpha}{\cos\alpha}$ ...(i)

and $\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} = \sqrt{\frac{(1-\sin\alpha)^2}{\cos^2\alpha}} = \left|\frac{1-\sin\alpha}{\cos\alpha}\right| = \frac{1-\sin\alpha}{|\cos\alpha|}$ ...(ii)

From (i) and (ii) two expressions are equal only if $\cos \alpha > 0$, i.e. $-\pi/2 < \alpha < \pi / 2$

$\therefore \sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}$ and $\sec \alpha - \tan \alpha$ are equal only

when $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example : 13 If $f(\theta)=1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$ then the maximum value of $f(\theta)$ is .

(1) 1 (2) 2 (3) 3 (4) 4

Solution : We have $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1+\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)+\sqrt{2}(\cos\theta+\sin\theta) = 1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right)(\cos\theta+\sin\theta)$

$= 1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right).\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$ The maximum value $1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right).\sqrt{2}=4$

Example : 14 The value of sin 20° sin 40° sin 60° sin 80° is

(1) 3/8 (2) 1/8 (3) 3/16 (4) none of these

Solution : sin 20° sin 40° sin 60° sin 80°

$= \frac{\sqrt{3}}{2}\sin 20^\circ\sin(60^\circ-20^\circ)\sin(60^\circ+20^\circ) = \frac{\sqrt{3}}{2}\sin 20^\circ\left(\sin^2 60^\circ-\sin^2 20^\circ\right)$

$= \frac{\sqrt{3}}{2}\sin 20^\circ\left(\frac{3}{4}-\sin^2 20^\circ\right) = \frac{\sqrt{3}}{8}(3\sin 20^\circ-4\sin^3 20^\circ) = \frac{\sqrt{3}}{8}\sin 60^\circ = \frac{\sqrt{3}}{8}.\frac{\sqrt{3}}{2}=\frac{3}{16}$

Example : 15 Value of sin600°

Solution : $\sin(3(180^\circ)+60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

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8. MULTIPLE ANGLES AND HALF ANGLES :

(1) sin 2A = 2 sinA cosA ; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(2) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$.

$2\cos^2 A = 1 + \cos 2A$, $2\sin^2 A = 1 - \cos 2A$; $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

$2\cos^2\frac{\theta}{2} = 1+\cos\theta$, $2\sin^2\frac{\theta}{2} = 1-\cos\theta$.

(3) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

(4) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(5) $\sin 3A = 3\sin A - 4\sin^3 A$

(6) $\cos 3A = 4\cos^3 A - 3\cos A$

(7) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

9. THREE ANGLES:

(i) sin (A + B + C) = sin A cos B cos C + sin B cos A cos C + sin C cos A cos B – sin A sin B sin C

(ii) $\cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(iii) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

DETECTIVE MIND

$\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \dfrac{S_1 - S_3 + S_5 - ......}{1 - S_2 + S_4 - ....}$

where S_i denotes sum of product of tangent of angles taken i at a time

10. CONDITIONAL IDENTITIES:

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\dfrac{A}{2} \cos\dfrac{B}{2} \cos\dfrac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\dfrac{A}{2} \sin\dfrac{B}{2} \sin\dfrac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\dfrac{A}{2} \tan\dfrac{B}{2} + \tan\dfrac{B}{2} \tan\dfrac{C}{2} + \tan\dfrac{C}{2} \tan\dfrac{A}{2} = 1$

(vii) $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2} . \cot\dfrac{B}{2} . \cot\dfrac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \dfrac{\pi}{2}$, then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

11. SINE AND COSINE SERIES:

(i) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \dfrac{n-1}{2}\beta\right)$

(ii) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... + \cos\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \dfrac{n-1}{2}\beta\right)$

where : $\beta \neq 2m\pi$, $m \in I$

12. PRODUCT SERIES OF COSINE ANGLES

$\cos\theta . \cos 2\theta . \cos 2^2\theta . \cos 2^3\theta ...... \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

13. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS:

(1) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $\theta \in R$

(2) Max. and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$

(3) If $f(\theta) = a\cos(\alpha + \theta) + b\cos(\beta + \theta)$ where a, b, α and β are known quantities then

$$-\sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)} \leq f(\theta) \leq \sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)}$$

(4) If A, B, C are the angles of a triangle then maximum value of

sinA + sinB + sinC and sinA sinB sinC occurs when A = B = C = 60^0

(5) In case a quadratic in sinθ or cosθ is given then the maximum or minimum values can be interpreted by making a perfect square.

SOLVED EXAMPLES

Example : 16 If cos 2x + 2 cos x = 1, then $\sin^2x$ ($2-\cos^2x$) is equal to

(1) 2 (2) 1 (3) 3 (4) 4

Solution : Here, cos 2x + 2 cos x = 1

$\Rightarrow$ $2\cos^2x-1+2\cos x-1=0$

$\Rightarrow$ $\cos^2x+\cos x-1=0$

or $\cos x = \dfrac{-1+\sqrt{5}}{2}$, neglecting $\left(\dfrac{-1-\sqrt{5}}{2}\right)$

as $-1 \le \cos x \le 1$ and $\left(\dfrac{-1-\sqrt{5}}{2}\right) < -1$

$\cos^2 x = \left(\dfrac{\sqrt{5}-1}{2}\right)^2 = \dfrac{6-2\sqrt{5}}{4} = 3-\sqrt{5}$

$\sin^2x\,(2-\cos^2x) = \left(1-\dfrac{3-\sqrt{5}}{2}\right)\left(2-\dfrac{3-\sqrt{5}}{2}\right) = \left(\dfrac{\sqrt{5}-1}{2}\right)\left(\dfrac{\sqrt{5}+1}{2}\right) = 1$

Example : 17 $\sin 67\dfrac{1}{2}^\circ + \cos 67\dfrac{1}{2}^\circ$ is equal to

(1) $\dfrac{1}{2}\sqrt{4+2\sqrt{2}}$ (2) $\dfrac{1}{2}\sqrt{4-2\sqrt{2}}$ (3) $-\dfrac{1}{2}\sqrt{4+2\sqrt{2}}$ (4) $\dfrac{1}{2}\sqrt{-4+2\sqrt{2}}$

Solution : $\sin 67\dfrac{1}{2}^\circ + \cos 67\dfrac{1}{2}^\circ = \sqrt{1+\sin 135^\circ} = \sqrt{1+\dfrac{1}{\sqrt{2}}}$ (using cosA + sinA = $\sqrt{1+\sin 2A}$)

$= \dfrac{1}{2}\sqrt{4+2\sqrt{2}}$ ....(i)

Example : 18 The value of $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$ is

(1) $\dfrac{1}{2}$ (2) $\cos\dfrac{\pi}{8}$ (3) $\dfrac{1}{8}$ (4) $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$

Solution : $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\left(\pi-\dfrac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\dfrac{\pi}{8}\right)\right)$

$= \left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{\pi}{8}\right) = \left(1-\cos^2\dfrac{\pi}{8}\right)\left(1-\cos^2\dfrac{3\pi}{8}\right)$

$= \frac{1}{4}\left(2-1-\cos\frac{\pi}{4}\right)\left(2-1-\cos\frac{3\pi}{4}\right) = \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)$

$= \left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$

Example : 19 The value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$ is

(1) 0 (2) 1 (3) $\frac{1}{2}$ (4) None of these

Solution : Use $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta)+\ldots+ \cos(\alpha+\overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\frac{2\alpha+(n-1)\beta}{2}$

Here $\alpha = \frac{\pi}{11}, \beta=\frac{2\pi}{11}, n=5$

$\Rightarrow \frac{\sin\left(\frac{10\pi}{11}\times\frac{1}{2}\right)}{\sin\left(\frac{2\pi}{11}\times\frac{1}{2}\right)}\cos\left(\frac{\frac{2\pi}{11}+\frac{8\pi}{11}}{2}\right) = \frac{\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$

$\Rightarrow \frac{1}{2}\left[\frac{\sin\left(\frac{10\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}\right] = \frac{1}{2}\left[\frac{\sin\left(\pi-\frac{\pi}{11}\right)}{\sin\frac{\pi}{11}}\right] = \frac{1}{2}\left[\frac{\sin\frac{\pi}{11}}{\sin\frac{\pi}{11}}\right] = \frac{1}{2}$

Example : 20 $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ equals to -

(1) 1/2 (2) 1/4 (3) 3/2 (4) 3/4

Solution : $= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$

$= 2\left(\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}\right) = \frac{1}{2}\left[\left(2\cos^2\frac{\pi}{8}\right)^2+\left(2\cos^2\frac{3\pi}{8}\right)^2\right]$

$= \frac{1}{2}\left[\left(1+\cos\frac{\pi}{4}\right)^2+\left(1+\cos\frac{3\pi}{4}\right)^2\right] = \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}[2+1] = \frac{3}{2}$

Example : 21 If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C =$

(1) $1 - 4\cos A\cos B\cos C$ (3) $4\sin A\sin B\sin C$

(3) $1 + 2\cos A\cos B\cos C$ (4) $1 - 4\sin A\sin B\sin C$

Solution : $\cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + \cos 2C$

$= 2\cos\left(\frac{3\pi}{2}-C\right)\cos(A - B) + \cos 2C \quad \because A + B + C = \frac{3\pi}{2}$

$= -2 \sin C \cos(A - B) + 1 - 2\sin^2 C = 1 - 2\sin C\,[\cos(A - B) + \sin C)$

$= 1 - 2\sin C\left[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)\right] = 1 - 2\sin C\,[\cos(A - B) - \cos(A + B)]$

$= 1 - 4 \sin A \sin B \sin C$

Example : 22 In any triangle ABC, sin A – cos B = cos C, then angle B is -

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

Solution : We have, sin A – cos B = cos C

sin A = cos B + cos C

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$\because\ A + B + C = \pi$

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right) \Rightarrow \cos\frac{A}{2} = \cos\frac{B-C}{2}$ or $A = B - C$

But $A + B + C = \pi$

Therefore $2B = \pi \Rightarrow B = \pi/2$

Example : 23 tan 9° – tan27° – tan 63° + tan 81° is equals to -

(1) 0 (2) 1 (3) –1 (4) 4

Solution : tan 9° + tan 81° – (tan 27° + tan 63°)

(tan 9° + cot 9°) – (tan 27° + cot 27°)

$= \left(\frac{\sin 9^\circ}{\cos 9^\circ}+\frac{\cos 9^\circ}{\sin 9^\circ}\right)-\left(\frac{\sin 27^\circ}{\cos 27^\circ}+\frac{\cos 27^\circ}{\sin 27^\circ}\right) = \frac{1}{\sin 9^\circ \cos 9^\circ}-\frac{1}{\cos 27^\circ \sin 27^\circ}$

$= \frac{2}{\sin 18} - \frac{2}{\sin 54^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\cos 36^\circ} = \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1} = 8\left[\frac{\sqrt{5}+1-\sqrt{5}+1}{(\sqrt{5}-1)(\sqrt{5}+1)}\right] = \frac{16}{4} = 4$