KGS IAS

Maths Optional

By Dhrub Singh Sir

and has the same sum as the original series.

Remarks: ① If Σu_n is a +ve term series converging to 's' then any rearrangement Σv_n of Σu_n also converges to 's'.

② If a +ve term series

Note: The rearrangement (derangement) or changing the order of terms in a series may not only alter the sum of the series but may change its nature all together.

Dirichlet's Th: A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely

→ The given series is a rearrangement of the series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots$$

This series is abs. cgt. (As $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ is cgt.).

∴ By Dirichlet's Th.

$\sum u_n$ is divergent then any rearrangement of $\sum u_n$, say $\sum v_n$ is also divergent.

③ An absolutely convergent series is also called unconditionally cgt.

Prob: Discuss the convergence of the series $1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}-\frac{1}{4^2}-\cdots$

Pringsheim's method

Let $f(n)$ be a +ve, monotonically decreasing function converging to zero.

Then by Leibnitz's test the series $\sum (-1)^{n-1} f(n)$ is cgt.

Let the terms of the series

The given series is abs. cgt. (cgt.)

Riemann's th: A conditionally convergent series can be made by an appropriate rearrangement of terms

(i) to converge to any real no.

(ii) to diverge to either $+\infty$ or $-\infty$.

(iii) to oscillate finitely or infinitely.

In particular, consider the series $\sum (-1)^{n-1} \cdot \frac{1}{n}$,

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - - = \log 2$$

Here $f(n) = \frac{1}{n}$

Also, $g = m \cdot f(m) = m \cdot \frac{1}{m} = 1$

If the terms of the series are rearranged s.t. α '+ve' terms succeeded by β '-ve' $\sum (-1)^{n-1} f(n)$ be rearranged s.t. α '+ve' terms succeeded by β '-ve' terms alternately.

Let $g = m \cdot f(m)$ & $K = \frac{\alpha}{\beta}$

then alteration in the sum due to this rearrangement is $\frac{1}{2} g \log K$

→ The given series is rearrangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

$$=\sum(-1)^{n-1}\cdot\frac{1}{n}=\log 2$$

In the rearranged series, one +ve term succeeded by two -ve terms.

$\alpha=1 \qquad \beta=2$

$$K=\frac{\alpha}{\beta}=\frac{1}{2}$$

terms alternately,

$$K=\frac{\alpha}{\beta}$$

Sum of the rearranged series $=\log 2+\frac{1}{2}\log K$

<u>Prob</u>: Find the sum of the series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\cdots$$

Prob: Find the sum of the series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$$

$\rightarrow$ $\alpha=2, \quad \beta=1$

$$K=\frac{\alpha}{\beta}=2$$

$$\text{Sum}=\log 2+\frac{1}{2}\log K$$

$$=\log 2+\frac{1}{2}\log 2$$

$$=\frac{3}{2}\log 2.$$

$$\delta=m\cdot f(m)$$

$$=m\cdot\frac{1}{m}=1$$

The sum of rearranged series $=\log 2+\frac{1}{2}\delta\log K$

$$=\log 2+\frac{1}{2}\log\frac{1}{2}$$

$$=\log 2+\frac{1}{2}\log 2^{-1}$$

$$=\log 2-\frac{1}{2}\log 2$$

$$=\frac{1}{2}\log 2$$

$= \log 2\cdot\sqrt{3}$

$= \log (12)^{\frac{1}{2}}$

$= \frac{1}{2}\log 12$

Prob: Find the sum of the series

$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+ - - -$

Prob: Find the sum of the series

$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}$

$+ - - -$

$\longrightarrow$

$\alpha = 3 \qquad \beta = 1$

$K = \frac{\alpha}{\beta} = 3$

$\text{Sum} = \log 2 + \frac{1}{2}\log 3$

$= \log 2 + \log\sqrt{3}$

prob: Investigate what de-arrangement of the series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

will reduce its sum to zero.

→ The given series $\sum (-1)^{n-1} \cdot \frac{1}{n}$ is conditionally cgt. and its sum is $\log 2$.

Let it be dearranged by taking alternately α '+ve' terms and β '-ve' terms, so that

→ $\alpha = 3, \quad \beta = 2$

$$K = \frac{\alpha}{\beta} = \frac{3}{2}$$

$$\text{Sum} = \log 2 + \frac{1}{2} \log \frac{3}{2}$$

$$= \log 2 + \frac{1}{2}\left[\log 3 - \log 2\right]$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \log 3$$

$$= \frac{1}{2}\left[\log 2 + \log 3\right]$$

$$= \frac{1}{2} \log 6.$$

$\Rightarrow \log K = \log 2^{-2} = \log \frac{1}{4}$

$\Rightarrow K = \frac{1}{4}$

$\alpha = 1, \quad \beta = 4$

To get the sum '0', the rearrangement be like one '+ve' term succeeded by four '-ve' term alternately.

$K = \frac{\alpha}{\beta} \qquad g = m \cdot f(m)$

$= m \cdot \frac{1}{m} = 1$

Sum of the rearranged series $= \log 2 + \frac{1}{2} g \log K$

$= \log 2 + \frac{1}{2} \log K$

$\log 2 + \frac{1}{2} \log K = 0$

$\frac{1}{2} \log K = -\log 2$

$\Rightarrow \log K = -2 \log 2$

$\Rightarrow \log 2 + \frac{1}{2}\log K = \frac{1}{2}\log 2$

$\Rightarrow \frac{1}{2}\log K = \frac{1}{2}\log 2 - \log 2$

$\Rightarrow \log K = \log 2 - 2\log 2$

$\Rightarrow \log K = \log 2 - \log 2^2 = \log \frac{\cancel{2}}{4_2}$

$K = \frac{1}{2}$

$\alpha = 1, \quad \beta = 2$

one +ve term followed by two -ve terms alternately.

$\therefore$ Rearranged series is

$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} \cdots\cdots$

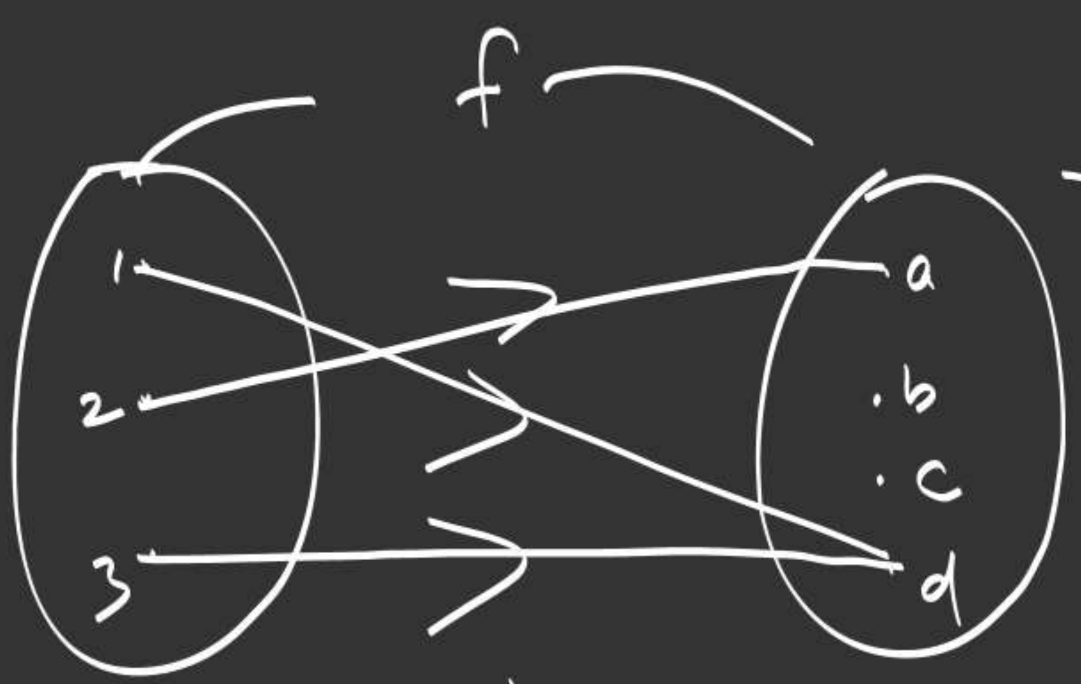
& sum $= 0$

<u>Prob:</u> Rearrange the series $\sum (-1)^{n+1} \cdot \frac{1}{n}$ to converge to $\frac{1}{2}\log 2$.

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\cdots$$

$$=\frac{1}{2}\ln 2$$

Limit and continuity



A (Domain)

B (Co-Domain)

$f(1) = d$

$f(2) = a$

$f(3) = d$

Range = set of image

$= \{a, d\} \subseteq$ Co-Domain.

<u>Function</u>: A function $f : A \to B$ is a rule which associate $\boxed{\text{each}}$ element of A to a $\boxed{\text{unique}}$ element of B.

$A = \{1, 2, 3\}$ $B = \{a, b, c, d\}$

→ $f : \mathbb{R} \to \mathbb{R}$

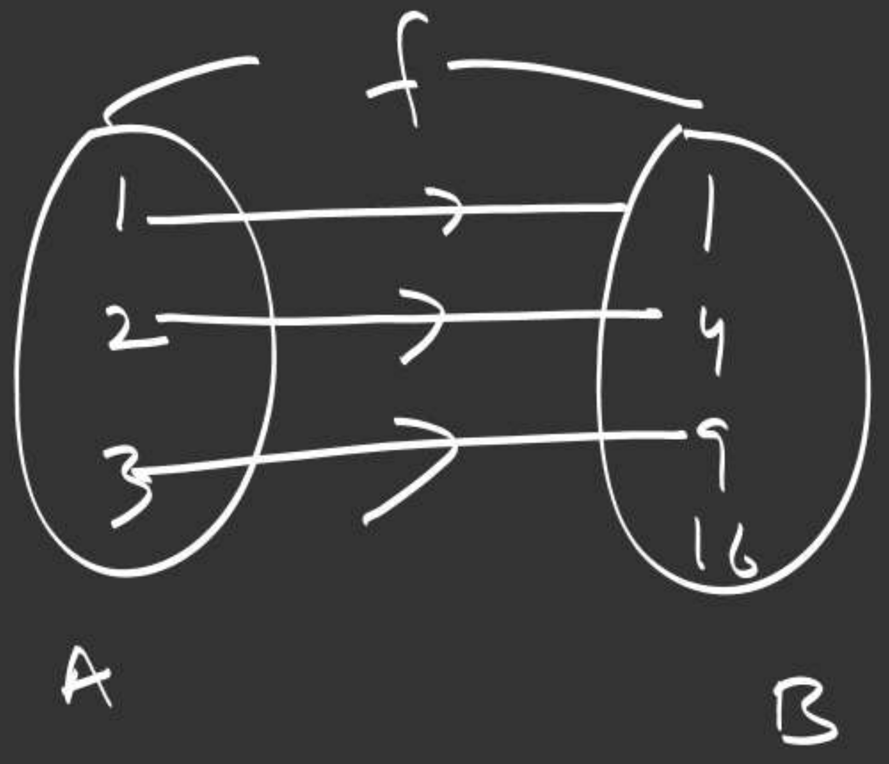
real valued function

→ $y = f(x)$ $\quad f : \underline{\mathbb{R} \to \mathbb{R}}$

Domain of fn. = set of value of x for which y is a real no.

Ex: $y = f(x) = \sqrt{x-1}$ $\quad$ Range $= [0, \infty[$

Dom $= [1, \infty[$



$f(1) = 1 = 1^2$

$f(2) = 4 = 2^2$

$f(3) = 9 = 3^2$

$f : A \to B$

$y = f(x) = x^2$

Some elementary function
(Real valued fn.)

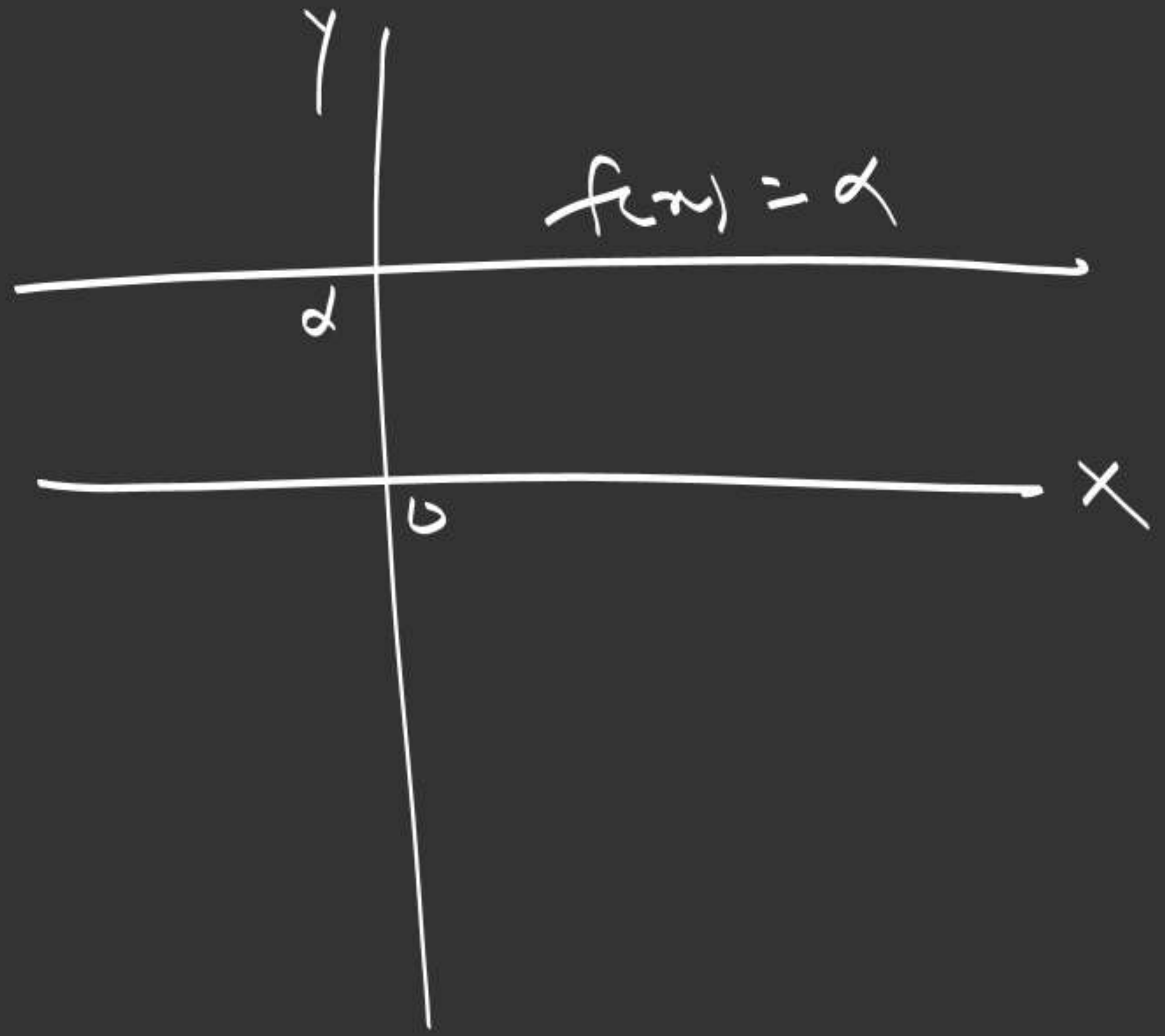
Constant function:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \underline{\alpha} \quad \forall x \in \mathbb{R}$$

Domain $= \mathbb{R}$

Range $= \{\alpha\}$



Identity fn: $f : \mathbb{R} \to \mathbb{R}$

$$y = f(x) = x.$$

Domain $=$ Range $= \mathbb{R}$

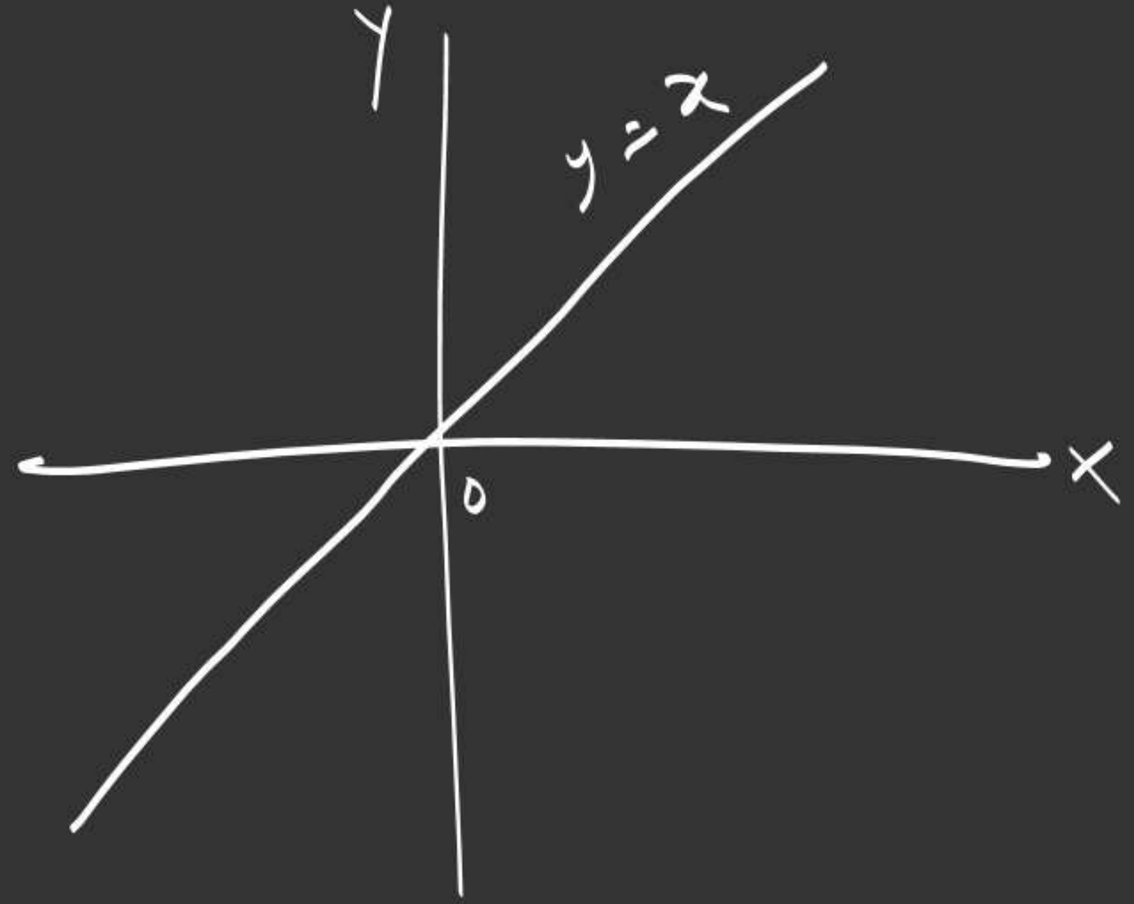
Rational fn:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \frac{p(x)}{\varphi(x)}.$$

$p(x)$, $\varphi(x)$ are poly fn.

$$\operatorname{Dom} f = \mathbb{R} \sim \{x : \varphi(x) = 0\}$$



Polynomial function:

$$f : \mathbb{R} \to \mathbb{R}$$

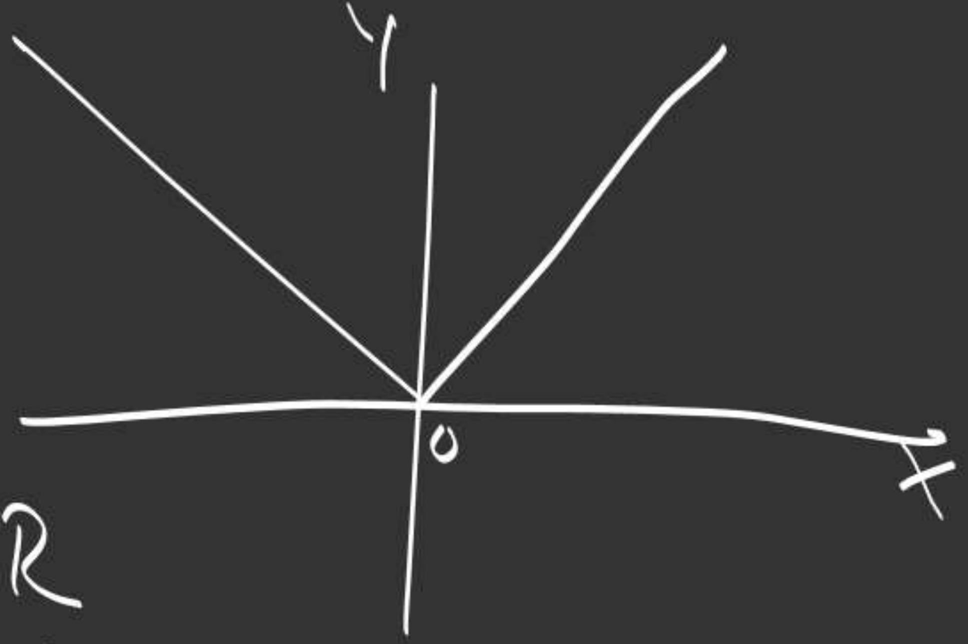
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$n \in \mathbb{N}$, $a_n \neq 0$ $\quad a_i \in \mathbb{R}$.

Modulus function:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Domain $= \mathbb{R}$

Range $= [0, \infty[$.

Ex: $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 4}$

$$\text{Dom} f = \mathbb{R} \setminus \{2, -2\}$$

Power function: $f : \mathbb{R}^+ \to \mathbb{R}$

$$f(x) = x^n, \quad n \in \mathbb{R}$$

Ex: $f(x) = x^{\frac{1}{2}}$.

Dom $= [0, \infty[$