KGS IAS

Maths Optional

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$$\rightarrow \left(2\sqrt{x}\,p + 3x - a\right)\left(2\sqrt{x}\,p - 3x + a\right) = 0$$

$$2\sqrt{x}\,p + 3x - a = 0$$

$$\Rightarrow 2\sqrt{x}\,p = a - 3x$$

$$\Rightarrow 2p = a\cdot x^{-\frac{1}{2}} - 3\sqrt{x}$$

$$\Rightarrow 2\frac{dy}{dx} = a\cdot x^{-\frac{1}{2}} - 3\sqrt{x}$$

solvable for p.

Prob: $4xp^2 = (3x-a)^2$

$$\rightarrow \quad 4xp^2 - (3x-a)^2 = 0 \quad \text{--- (1)}$$

i.e. $\left(2\sqrt{x}\,p\right)^2 - (3x-a)^2 = 0$

Gen. solⁿ:

$$(y+c)^2 = x(x-a)^2$$

$$2\sqrt{x}\,p = 3x - a$$

$$y + c = x^{3/2} - a\cdot x^{1/2}$$

$\rightarrow$ Soln.

$$\left[y + c - \left(a x^{1/2} - x^{3/2}\right)\right]\left[y + c + \left(a x^{1/2} - x^{3/2}\right)\right] = 0$$

$$2\,dy = a\cdot x^{-1/2}\,dx - 3\sqrt{x}\,dx$$

Int.

$$2y = \frac{a\cdot x^{1/2}}{\frac{1}{2}} - \cancel{3}\frac{x^{3/2}}{\cancel{3}/2} + \text{const}$$

$$\Rightarrow 2y = 2a x^{1/2} - 2x^{3/2} + \text{const}$$

$$\Rightarrow y = a x^{1/2} - x^{3/2} + \text{const}$$

$$\Rightarrow y + c = a x^{1/2} - x^{3/2}$$

$$(y+c)^2-\left(ax^{\frac{1}{2}}-x^{\frac{3}{2}}\right)^2=0$$

$$\Rightarrow (y+c)^2=\left(ax^{\frac{1}{2}}-x^{\frac{3}{2}}\right)^2$$

$$=a^2x+x^3-2ax^2$$

$$=x\left[a^2+x^2-2ax\right]$$

i.e. $$(y+c)^2=x(x-a)^2$$

$\rightarrow p^2 \cdot x^4 - px - y = 0$ —— (1)

Gen. sol^n:

$$y + \frac{c}{x} = c^2$$

$$\Rightarrow \frac{xy + c}{x} = c^2$$

$$\Rightarrow x \cdot c^2 - c - xy = 0 \quad \text{—— (11)}$$

p-disc:

$$x^2 + 4x^4 y = 0$$

$$\underbrace{x^2}(1 + 4x^2 y) = 0 \quad \text{—— (111)}$$

Solvable for y

prob: Find the complete primitive and also examine the existence of singular sol^n. for the following eq^n:

$$y + px = p^2 x^4$$

Now

L.H.S. of (1)

$$= \left(\frac{1}{2x^3}\right)^2 \cdot x^4 - \frac{1}{2x^3} \cdot x + \frac{1}{4x^2}$$

$$= \frac{1}{4x^{\cancel{6}\,2}} \cdot \cancel{x^4} - \frac{1}{2x^2} + \frac{1}{4x^2}$$

$$= \frac{1}{4x^2} - \frac{1}{2x^2} + \frac{1}{4x^2}$$

$$= \frac{1-2+1}{4x^2} = 0$$

c-disc:

$$1+4x^2y=0 \quad \text{--- (1V)}$$

$1+4x^2y=0$ is a common factor of p-disc. & c-disc.

$$y = -\frac{1}{4x^2}$$

$$p = \frac{dy}{dx} = +\frac{1}{\cancel{4}_{2}} \times (\cancel{+2}) \cdot \frac{1}{x^3} = \frac{1}{2x^3}$$

Prob: $xp^2 - 2yp + ax = 0$ — (I)

→ General sol^n.

$$c^2x^2 - 2cy + a = 0 \quad \text{— (II)}$$

→ p-disc:

$$4y^2 - 4x \cdot ax = 0$$

$$\Rightarrow 4y^2 - 4ax^2 = 0$$

$$\Rightarrow y^2 - ax^2 = 0 \quad \text{— (III)}$$

$\therefore 1 + 4x^2 y = 0$

satisfies D.E.

$\therefore 1 + 4x^2 y = 0$ is the singular sol^n.

Extraneous locus in p-disc.

$\underline{x = 0}$ (Tac-locus)

It doesn't satisfy the D.E.

$\cancel{p}yp = \cancel{p}ax$

$p = \frac{ax}{y}$

$y^2 - ax^2 = 0$ satisfies the D.E. (1).

$\therefore y^2 - ax^2 = 0$ is the S.S.

c-disc

$4y^2 - 4ax^2 = 0$

$\Rightarrow y^2 - ax^2 = 0$ —— (IV)

$y^2 - ax^2 = 0$ is the common factor of p-disc. & c-disc.

$y^2 - ax^2 = 0$

$2yp - 2ax = 0$

$\Rightarrow y = c^3 + cx^2 - 2c^2x$

$\Rightarrow c^3 - 2c^2x + cx^2 - y = 0$ — (II)

p-desc: (I) $\Rightarrow$

$3p^2 - 4xy = 0$ [partially diff. (I) w.r.t 'p']

$\Rightarrow p^2 = \frac{4}{3}xy$ — (III)

From (I)

$p\left[p^2 - 4xy\right] = -8y^2$

Squaring.

Solvable for x

prob: solve and examine for singular solutions of the following D.E.

$p^3 - 4xyp + 8y^2 = 0$ — (I)

$\rightarrow$ General soln.

$y = c(c-x)^2$

$\Rightarrow y = c\left[x^2 + c^2 - 2cx\right]$

$$y^4 - \frac{4}{27}x^3y^3 = 0$$

$$\Rightarrow y^3\left[y - \frac{4}{27}x^3\right] = 0$$

$$y^2 \cdot \underline{y}\left[y - \frac{4}{27}x^3\right] = 0 \quad \text{— (VII)}$$

$\leftarrow$ disc.

$$3c^2 - 4cx + x^2 = 0$$

[part. diff. (II) w.r.t. c']

$$\Rightarrow 3c^2 - 3cx - cx + x^2 = 0$$

$$\Rightarrow 3c(c-x) - x(c-x) = 0$$

$$\Rightarrow (c-x)(3c-x) = 0$$

$$p^2\left[p^2 - 4xy\right]^2 = 64y^4$$

$$\Rightarrow \frac{4}{3}xy\left[\frac{4}{3}xy - 4xy\right]^2 = 64y^4$$

using (III)

$$\Rightarrow \frac{4}{3}xy\left[-\frac{8}{3}xy\right]^2 = 64y^4$$

$$\Rightarrow \frac{4}{3}xy \times \frac{64}{9}x^2y^2 = \cancel{64}y^4$$

$$\Rightarrow \frac{4}{27}x^3y^3 = y^4$$

$$\frac{x^3}{27} - \frac{2x^3}{9} + \frac{x^3}{3} - y = 0$$

$$\Rightarrow \frac{x^3 - 6x^3 + 9x^3 - 27y}{27} = 0$$

$$\Rightarrow 4x^3 - 27y = 0$$

$$\Rightarrow y = \frac{4}{27}x^3$$

$$\Rightarrow y - \frac{4}{27}x^3 = 0 \quad \text{——} \quad (V)$$

$$c = x, \ \frac{x}{3}$$

when $c = x$ (put in (1))

$$\cancel{x^3} - \cancel{2x^3} + \cancel{x^3} - y = 0$$

$$y = 0 \quad \text{——} \quad (IV)$$

when $c = \frac{x}{3}$ (put in (1))

$$\left(\frac{x}{3}\right)^3 - 2\left(\frac{x}{3}\right)^2 \cdot x + \frac{x}{3} \cdot x^2 - y = 0$$

Clairaut's form

Prob: Find the complete primitive and singular soln. of

$$y = px + \sqrt{b^2 + a^2p^2}$$

→ Gen soln: $y = cx + \sqrt{b^2 + a^2c^2}$

→ $y - px = \sqrt{b^2 + a^2p^2}$

Squaring both sides

from (IV) and (V)

C-disc:

$$y\left(y - \frac{4}{27}x^3\right) = 0 \quad \text{— (VI)}$$

From (VI) & (VII)

$y = 0$, $y = \frac{4}{27}x^3$ are the common factors.

$\underbrace{y = 0, \quad y = \frac{4}{27}x^3}_{S.S.}$ satisfy D.E

$$\Rightarrow -b^2x^2 - a^2y^2 + a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Common to both p-disc & c-disc.

It satisfies D.E.

(check it!)

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (S.S.)$$

$$(y - px)^2 = b^2 + a^2p^2$$

$$\Rightarrow p^2x^2 + y^2 - 2pxy = b^2 + a^2p^2$$

$$\Rightarrow (x^2 - a^2)p^2 - 2pxy + y^2 - b^2 = 0$$

<u>p-disc</u> (<u>c-disc.</u>)

$$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$\Rightarrow \cancel{4x^2y^2} - 4\left[\cancel{x^2y^2} - b^2x^2 - a^2y^2 + a^2b^2\right] = 0$$

→ Given that

$$(px-y)(x-py)=2p$$

its general solⁿ.

is $y^2 = cx^2 - \dfrac{2c}{1-c}$.

$$\rightarrow y^2 = \frac{c(1-c)x^2 - 2c}{1-c}$$

Reducible to clairaut's form

Prob: Reduce the D.E. $(px-y)(x-py)=2p$ to clairaut's form by the substitution $u=x^2$ and $v=y^2$ and find its complete primitive and its singular solⁿ. if any.

Given D.E. is

$$(px-y)(x-py)=2p$$

$$\Rightarrow px^2-p^2xy-xy+py^2=2p$$

$$\Rightarrow -p^2xy+p(x^2+y^2)-xy=2p$$

$$\Rightarrow -p^2xy+p(x^2+y^2-2)-xy=0$$

$$\Rightarrow p^2xy-p(x^2+y^2-2)+xy=0$$

p-disc.

$$(x^2+y^2-2)^2-4x^2y=0 \quad \text{--- (ii)}$$

$$(1-c)y^2=c(1-c)x^2-2c$$

$$\Rightarrow y^2-cy^2=cx^2-c^2x^2-2c$$

$$\Rightarrow c^2x^2-c\left[x^2+y^2-2\right]+y^2=0$$

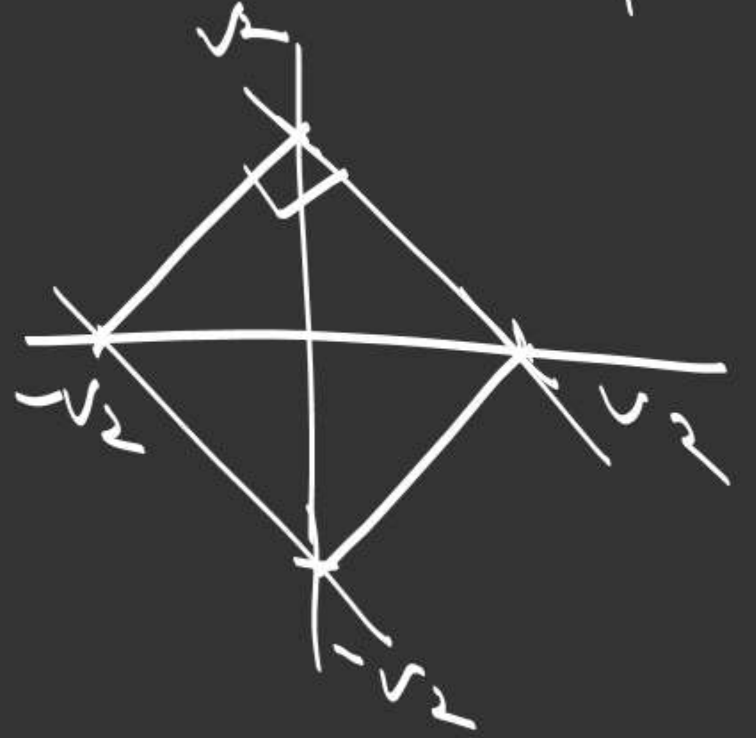
c-disc:

$$\left[x^2+y^2-2\right]^2-4x^2y^2=0 \quad \text{--- (i)}$$

Sing. solⁿ:

$x+y+\sqrt{2}=0$
$x+y-\sqrt{2}=0$
$x-y+\sqrt{2}=0$
$x-y-\sqrt{2}=0$

} Satisfy D.E (check it!)



(I) & (II) are identical

common factor:

$$(x^2+y^2-2)^2-(2xy)^2=0$$

$$\Rightarrow [x^2+y^2-2+2xy][x^2+y^2-2-2xy]=0$$

$$\Rightarrow [(x+y)^2-(\sqrt{2})^2][(x-y)^2-(\sqrt{2})^2]=0$$

$$\Rightarrow (x+y+\sqrt{2})(x+y-\sqrt{2})(x-y+\sqrt{2})(x-y-\sqrt{2})=0$$

→ Gen. soln:

$$c^2x^2 - c(x^2+y^2-1) + y^2 = 0$$

S.S:

$$x+y+1=0$$
$$x+y-1=0$$
$$x-y+1=0$$
$$x-y-1=0$$

(HW) prob: Reduce the eqn.

$$xyp^2 - (x^2+y^2-1)p + xy = 0$$

to clairaut's form by the substitution $u = x^2$ and $v = y^2$. Hence show that the eqn. represent a family of conics touching the four sides of a square.

$\rightarrow$

$xy = cy + c^2$

$\Rightarrow c^2 + cy - xy = 0$

c-desc. (p-dic.)

$y^2 + 4xy = 0$

$\Rightarrow y(y + 4x) = 0$

S.S.

$y = 0$	$y + 4x = 0$
$p = 0$	$p = -4$
satisfies D.E.	satisfies D.E

$\rightarrow$ 2023 prob: Reduce the eqn.

$x^2p^2 + y(2x+y)p + y^2 = 0$

to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the eqn. and show that $y + 4x = 0$ is a S.S. of the D.E.

2022

HW prob: Find the general and singular soln. of the D.E.

$$(x^2-a^2)p^2 - 2xyp + y^2 + a^2 = 0.$$

Also give the geometrical soln. b/w the general & singular soln.