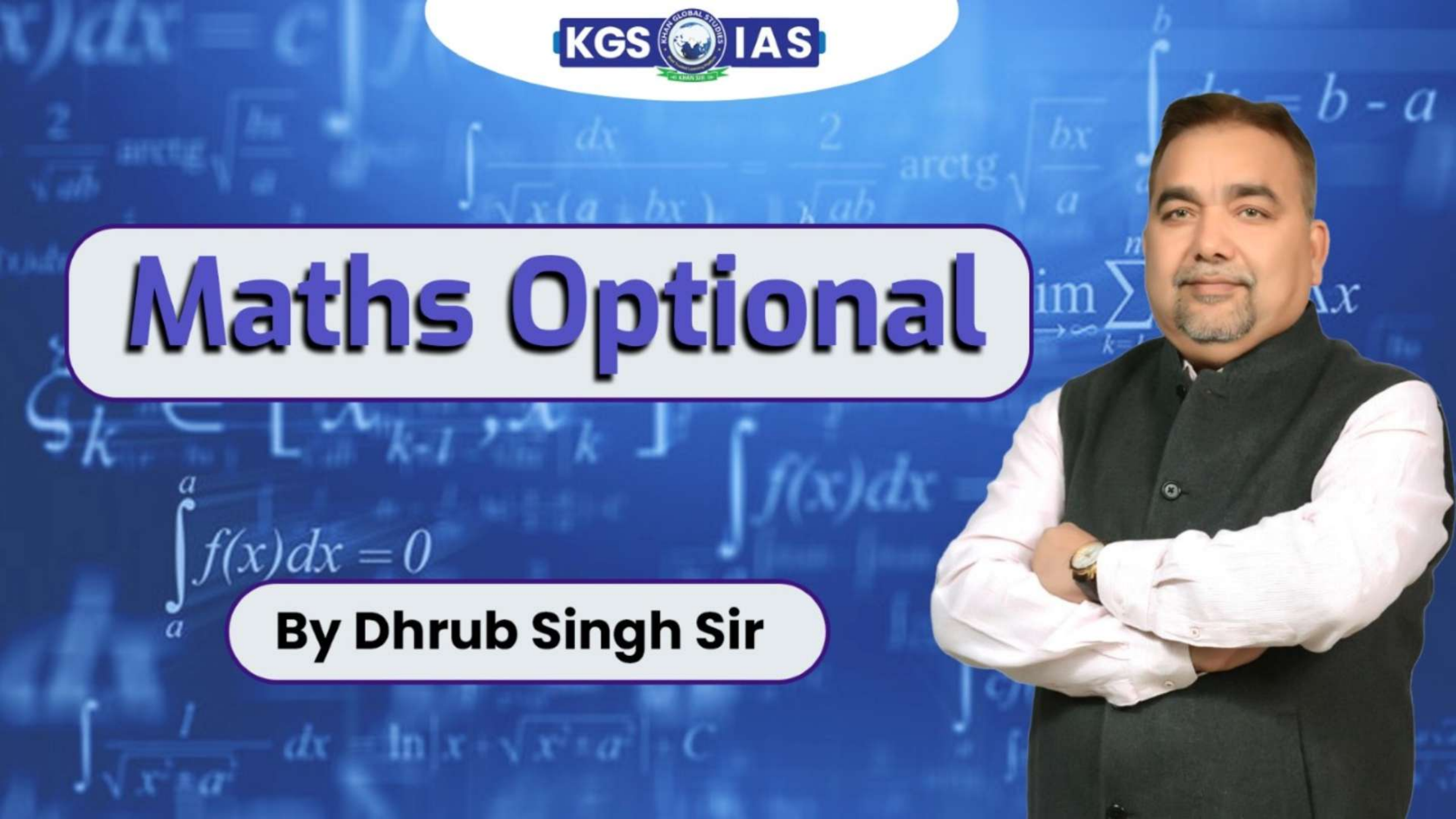
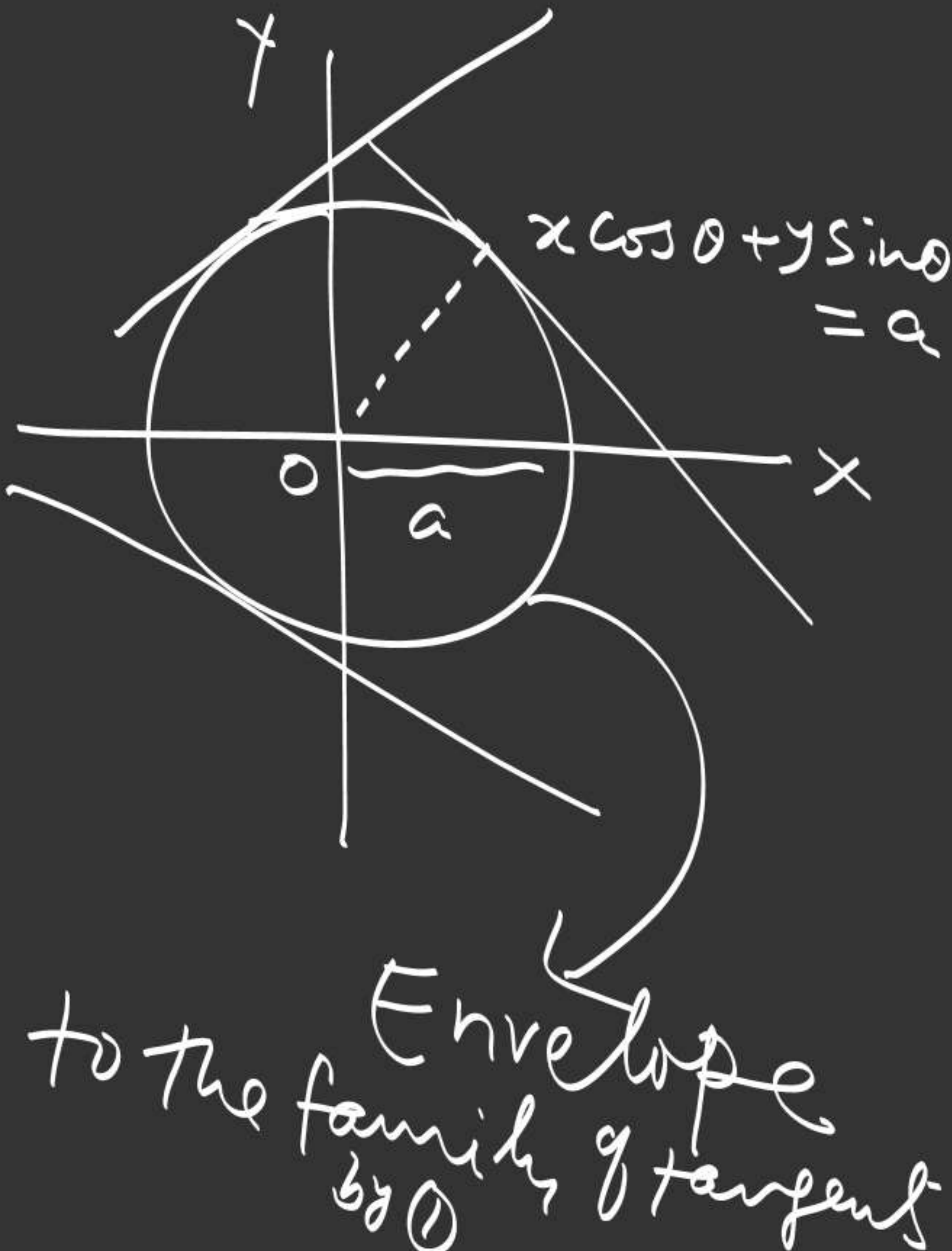

KGS IAS
Maths Optional
By Dhrub Singh Sir

$x\cos\theta + y\sin\theta = a$ — (1)



Singular Solⁿ.

Envelope.

$$x^2+y^2=a^2$$

At (x_1, y_1) tangent

$$xx_1+yy_1=a^2$$

At $(a\cos\theta, a\sin\theta)$

$$x\cdot a\cos\theta + y\cdot a\sin\theta = a^2$$

Differentiating this w.r.t 'x', we have

$$2y \cdot \frac{dy}{dx} = 4a$$

$$\Rightarrow 2yp = 4a$$

$$\Rightarrow p = \frac{4a}{2y} = \frac{2a}{y}$$

Put this in R.H.S. of (1)

$$R.H.S. = px + \frac{a}{p}$$

Ex: $y = px + \frac{a}{p}$ —— (1)

Clairaut's form. (11)

$y = mx + \frac{a}{m}$, m-arbitrary.

General solⁿ. of (1)

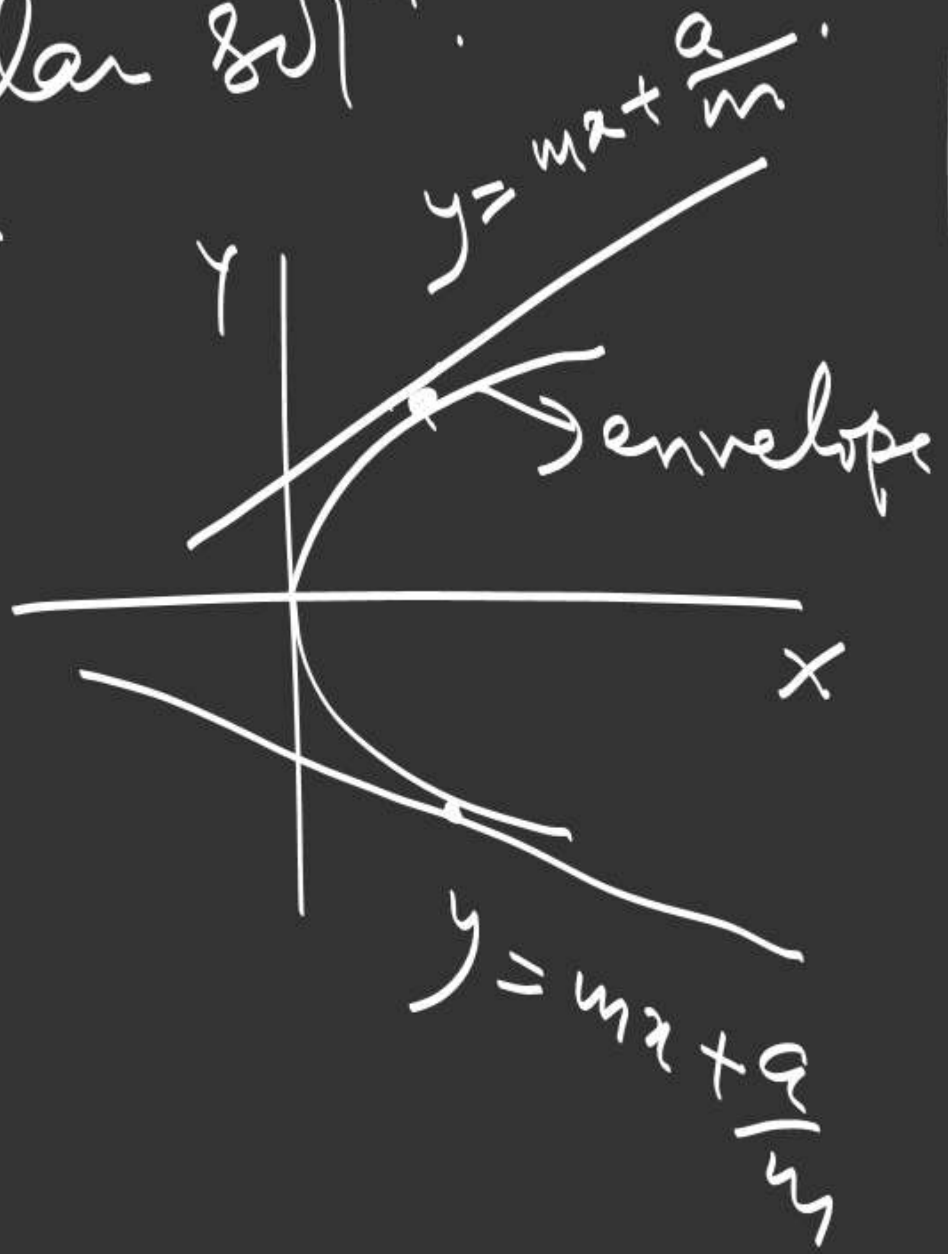
$y = mx + \frac{a}{m}$ is a tangent to the parabola

$$y^2 = 4ax$$

$\therefore\ y^2 = 4ax$ is a solⁿ. of (1)

+ singular solⁿ.

(Envelope of general solⁿ. of (1))



$$= \frac{2a}{y}\times x + \frac{a}{2a/y}$$

$$= \frac{2a}{y}\times x + \frac{ay}{2\cancel{a}}$$

$$= \frac{2ax}{y} + \frac{y}{2}$$

$$= \frac{\cancel{2a}}{\cancel{y}}\times \frac{y^2}{\cancel{4a}_{\,2}} + \frac{y}{2} = \frac{y}{2} + \frac{y}{2} = y$$

$= L.H.S$

rameter family of curves.

<u>Singular sol^n:</u> A solution of the diff. eqn. which is not derived from the general sol^n. by giving the particular value to the arbitrary constant, is called

<u>Envelope:</u> A curve which touches each member of a one parameter family of curves and at each point is touched by some member of the family of curves, is called the envelope of that one pa-

Eliminate p b/w (I) & (II),

p-discriminant $= 0$
(Relⁿ. obtained)

p-discriminant → locus (curve) for each point of which $f(x, y, p) = 0$ has equal values of p.

the singular solⁿ. The singular solⁿ. does n't involve any arbitrary constant.

p-discriminant

$$f(x, y, p) = 0 \quad \text{——— (I)}$$

$$\frac{\partial f}{\partial p} = 0 \quad \text{——— (II)}$$

c-disc.

locus (curve), for each point of which

$$\phi(x, y, c) = 0$$

has equal values of c.

c-discriminant

$$f(x, y, p) = 0$$

its general soln. is

$$\phi(x, y, c) = 0 \quad \text{--- (i)}$$

$$\frac{\partial \phi}{\partial c} = 0 \quad \text{--- (ii)}$$

Eliminate 'c' b/w (i) & (ii)

(ii) Whenever $\phi(x, y, c) = 0$, the general sol^n. of the D.E. $f(x, y, p) = 0$; possesses an envelope, the equation of envelope is the singular sol^n. of the D.E.

Note (1)

$$Ap^2 + Bp + C = 0$$

p-dis. $\quad B^2 - 4AC = 0$

General sol^n.

$$Ac^2 + Bc + K = 0$$

c-disc. $\quad B^2 - 4AK = 0$

Ex: Show that $x=0$ is the singular sol^n. of $4xp^2=(3x-a)^2$.

Sol^n: $4xp^2-(3x-a)^2=0$ — (1)

General sol^n. is

$(y+c)^2 = x(x-a)^2$

$\Rightarrow c^2+2cy+y^2-x(x-a)^2=0$ — (ii)

Working rule for finding the singular sol^n.

D.Eq^n: $f(x,y,p)=0$ —— (1)

$E(x,y)=0$ — singular sol^n. of (1) if it is a common factor of p-disc. & c-disc.

Also $E(x,y)$ must satisfy the D.E. (1).

$\Rightarrow 4\cancel{y^2} - 4\cancel{y^2} + 4x(x-a)^2 = 0$

$\Rightarrow 4x(x-a)^2 = 0$

i.e. $\underline{x}(x-a)^2 = 0$ —— (IV)

$x = 0$ is the common factor of (III) & (IV).

$x = 0$, $\frac{dx}{dy} = 0$ i.e. $\frac{1}{p} = 0$

From (I),

p-disc:

$0 + 4 \cdot 4x \cdot (3x-a)^2 = 0$

$\Rightarrow 16x(3x-a)^2 = 0$

$\Rightarrow \underline{x}(3x-a)^2 = 0$ —— (III)

From (II),

c-disc:

$(2y)^2 - 4\left[y^2 - x(x-a)^2\right] = 0$

Extraneous Loci: Each discriminant may have other factor which correspond to other loci associated with the general solⁿ of the given diff. eqn. Generally equations of these loci do not satisfy

From ①,

$$\frac{4x}{(3x-a)^2} = \frac{1}{p^2}$$

$\therefore x=0$ & $\frac{1}{p}=0$ satisfy them.

$\therefore x=0$ satisfies the D.E. ①

So, $x=0$ is the singular solⁿ of ①

Ex. (previous)

p-dis:

$$\underbrace{x(3x-a)^2}_{\text{Tac-locus.}}=0$$

c-dis:

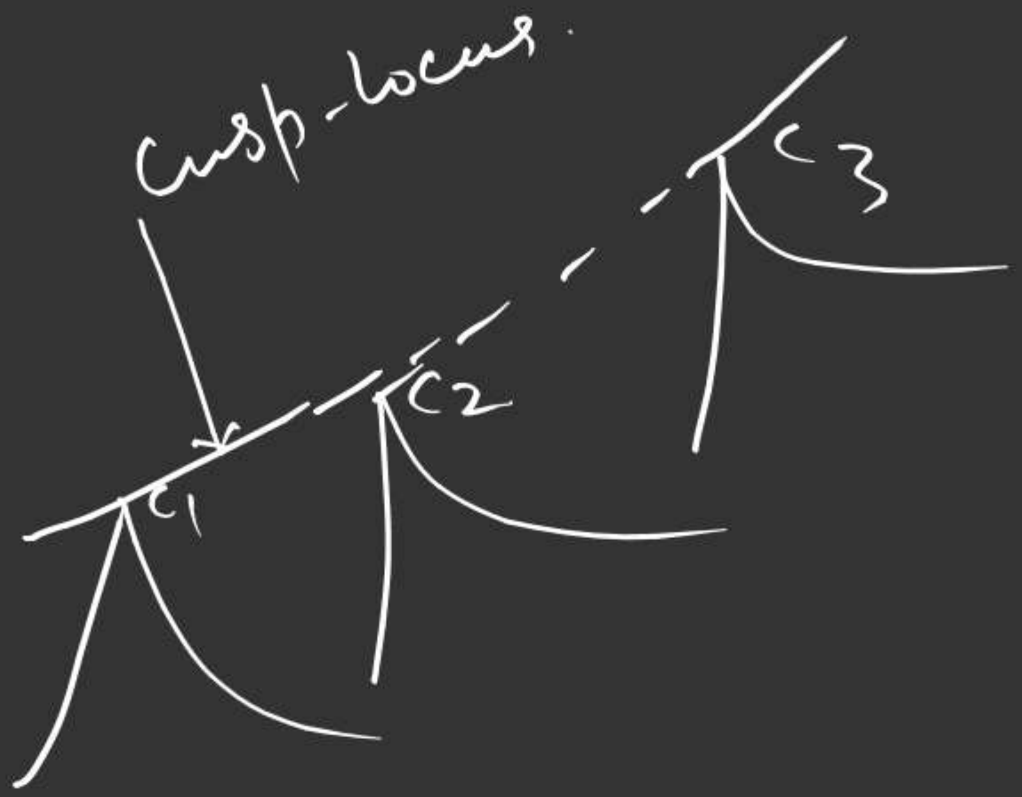
$$\underbrace{x(x-a)^2}_{\text{Node-locus.}}=0$$

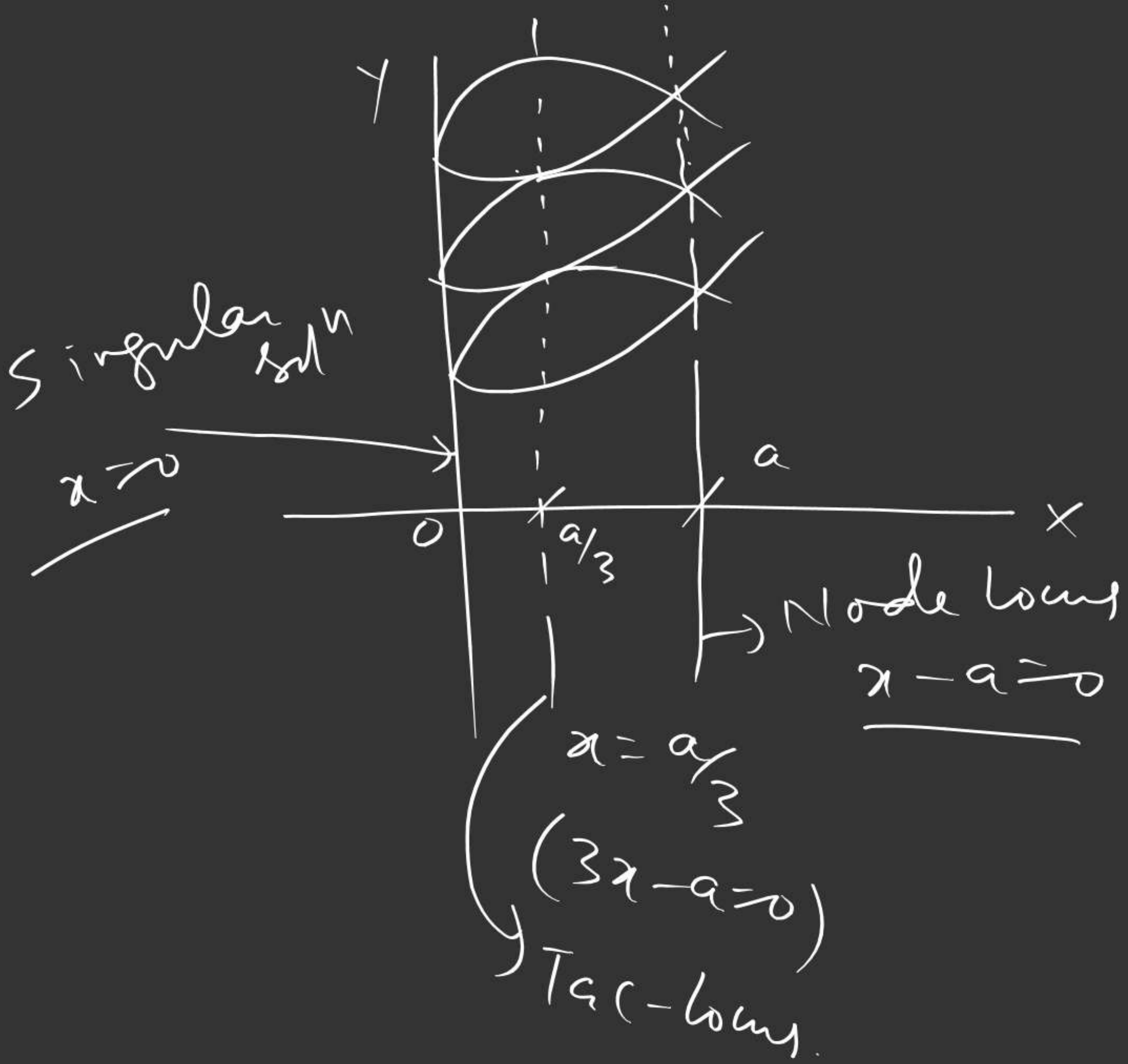
Gen soln:

$$(y+c)^2 = x(x-a)^2$$

the D.E.

	Tac-locus	Node locus	cusp-locus
p-disc.	✓	×	✓
C-disc	×	✓	✓


Cusp-locus.
c_1
c_2
c_3


Singular soln
$x=0$
Y
X
0
$a/3$
a
Node locus
$x-a=0$
$x=a/3$
$(3x-a=0)$
Tac-locus.

Step 2: p-disc. may include as a factor

(a) Envelope i.e. singular sol^n once (E)

(b) cusp-locus once (C)

(c) Tac-locus squared (T^2)

$$p\text{-disc} = ECT^2$$

Working rule for finding singular sol^n (Envelope) and extraneous loci

$$f(x, y, p) = 0$$

Step (1) (a) Find general sol^n.

$$\phi(x, y, c) = 0$$

(b) p-disc.

(c) c-disc.

Dirⁿ: obtain the complete primitive (gen. solⁿ.) and singular solⁿ. of the following equations. Explain the geometrical significance of the irrelevant factors that present themselves.

Step 3: c-disc. may include as a factor.

(a) Envelope i.e. singular solⁿ. once (E)

(b) cusp-locus cubed (C^3)

(c) Node-locus squared (N^2)

c-disc. $= EN^2C^3$

$$c^2 - 2yc + y^2 - \frac{4}{9}x(x-3a)^2 = 0 \quad \text{(II)}$$

p-disc: From (I)

$$0^2 + 4x(x-a)^2 = 0$$

$$x(x-a)^2 = 0 \quad \text{(III)}$$

c-disc: From (II)

$$\cancel{4y^2} - 4\left[\cancel{y^2} - \frac{4}{9}x(x-3a)^2\right] = 0$$

Solvable for p.

prob: $x p^2 = (x-a)^2$

$$\longrightarrow \quad x p^2 - (x-a)^2 = 0 \quad \text{(I)}$$

$$\left[\sqrt{x}\,p - (x-a)\right]\left[\sqrt{x}\,p + x - a\right] = 0$$

Gen. sol^n:

$$(y-c)^2 = \frac{4}{9}x(x-3a)^2$$

$\therefore x=0$ — singular soln. of (1)

From (III)

$x-a=0$ — Tac-locus.

Also $x=a$ doesn't satisfy (1)

From (IV) $x-3a=0$, Node-locus.

Also $x=3a$ doesn't satisfy (1).

$$\frac{16}{9}x(x-3a)^2=0$$

$$\Rightarrow x(x-3a)^2=0 \quad \text{——— (IV)}$$

$x=0$ is the common factor of (III) & (IV)

Also $x=0$, $\frac{dx}{dy}=0$ i.e. $\frac{1}{p}=0$

satisfy (1)

i.e. $\frac{x}{(x-a)^2}=\frac{1}{p^2}$