KGS IAS

Maths Optional

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→ The smallest limit point of bounded seq. $\langle a_n \rangle$ is called the limit inferior of the seq. $\langle a_n \rangle$.

$\liminf a_n$ or $\underline{\lim}\, a_n$

<u>Note</u>: $\liminf a_n \le \limsup a_n$

<u>Ex ①</u> $\langle (-1)^n \rangle$

Let $a_n = (-1)^n$

<u>Limit superior and limit inferior:</u>

→ The greatest limit point of bounded seq $\langle a_n \rangle$ is called the limit superior of the seq. $\langle a_n \rangle$

$\limsup a_n$ or $\overline{\lim}\, a_n$

$\langle 0, 2, 0, 2, \ldots \rangle$.

$\liminf a_n = 0 \quad \limsup a_n = 2$

Ex 3: $\left\langle \frac{1}{n} \right\rangle$

Let $a_n = \frac{1}{n}$

$\left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$

Set of limit ~~points~~ $= \{0\}$.

$\langle -1, 1, -1, 1, -1, 1, \ldots \rangle$

set of limit ~~points~~ $= \{-1, 1\}$

$\liminf a_n = -1$

$\limsup a_n = 1$.

Ex-2 $\langle 1 + (-1)^n \rangle$

Let $a_n = 1 + (-1)^n$

(iii) If a seq. $\langle a_n \rangle$ which is bdd below but not bdd above and has no other limit point than $+\infty$ then

$$\liminf a_n = \limsup a_n = +\infty$$

(iv) If a seq. $\langle a_n \rangle$ which is bdd above but not bdd below and has no other

$$\limsup a_n = \liminf a_n = 0$$

Remark: (i) If a seq. $\langle a_n \rangle$ is not bdd above we define

$$\limsup a_n = +\infty$$

(ii) If a seq. $\langle a_n \rangle$ is not bounded below, we define $\liminf a_n = -\infty$.

Ex: $\langle -n^2 \rangle$

Let $a_n = -n^2$

$\langle -1^2, -2^2, -3^2, \ldots \rangle$

$\limsup a_n = \liminf a_n$
$= -\infty$.

Ex: $\langle 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots \rangle$

$\limsup a_n = +\infty$

$\liminf a_n = 0$.

limit point than $-\infty$.

$\liminf a_n = \limsup a_n = -\infty$.

Ex: $\langle n^2 \rangle$

Let $a_n = n^2$

$\langle 1^2, 2^2, 3^2, \ldots \rangle$

$\limsup a_n = \liminf a_n = +\infty$.

Result (2): For a bdd seq. $\langle a_n \rangle$, show that $\liminf a_n = l$ iff for each $\varepsilon > 0$

(i) $a_n < l + \varepsilon$ for infinitely many values of n.

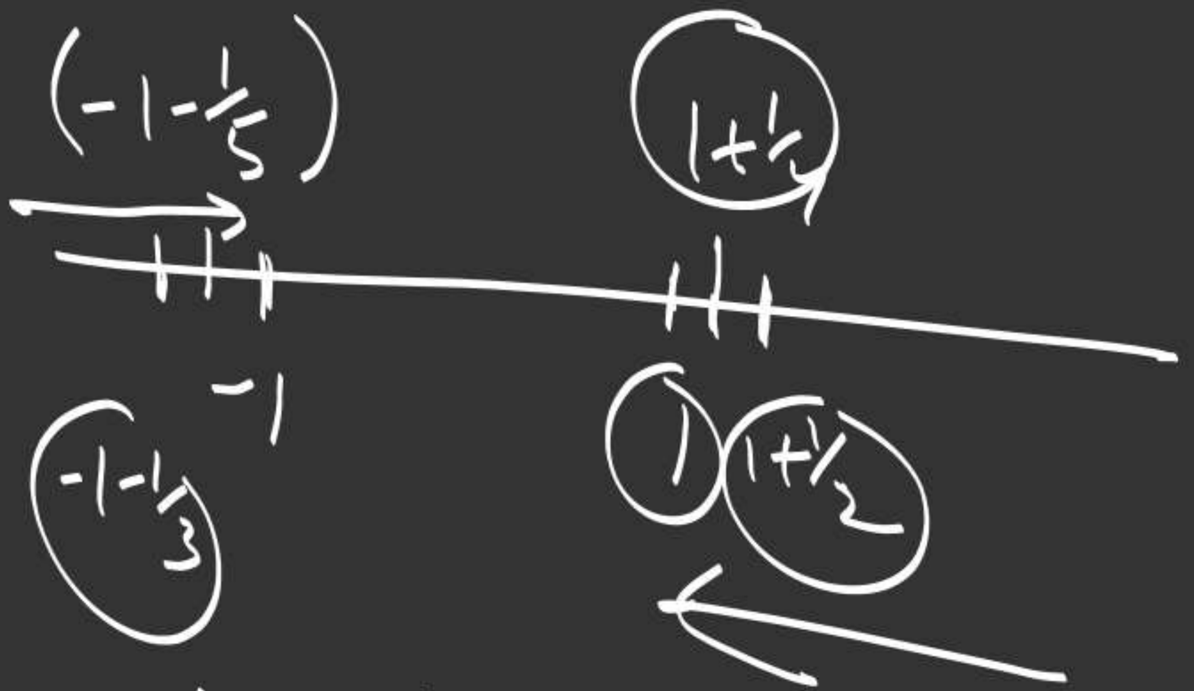
(ii) $a_n > l - \varepsilon$ for all except finitely many values of n.

Result (1): For a bdd seq. $\langle a_n \rangle$, show that $\limsup a_n = l$ iff for each $\varepsilon > 0$

(i) $a_n > l - \varepsilon$ for infinitely many values of n.

(ii) $a_n < l + \varepsilon$ for all except finitely many values of n.

$$\left\langle -(1+1),\ (1+\tfrac{1}{2}),\ -(1+\tfrac{1}{3}),\ (1+\tfrac{1}{4}),\ -(1+\tfrac{1}{5}) \cdots \right\rangle$$



set of limit points $= \{-1, 1\}$

$\limsup a_n = 1$

$\liminf a_n = -1.$

Result (3) A bdd seq. converges to l iff $\liminf a_n = \limsup a_n = l$.

Prob: Find the limit superior and the limit inferior of the following sequences:

(i) $\left\langle (-1)^n \left(1+\frac{1}{n}\right) \right\rangle$

(11) $\langle n(1+(-1)^n) \rangle$

$\langle 0, 4, 0, 8, 0, 12, \dots \rangle$

$\liminf a_n = 0$

$\limsup a_n = +\infty$.

Ex: (i) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$$= \sum \frac{1}{n}$$

(ii) $\sum \frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

Infinite series

$\rightarrow \quad \langle u_n \rangle$

$$\downarrow$$

$$\sum_{n=1}^{\infty} u_n = \underbrace{u_1+u_2+u_3+\cdots}_{\text{infinite series}}$$

$$\downarrow$$

$$\sum u_n$$

$\langle s_n \rangle$ —— S.O.P.S.

$$\boxed{\sum u_n \ \& \ \langle s_n \rangle \text{ have same nature.}}$$

Note: $\langle s_n \rangle \to u$

then $\sum u_n = u$

Sequence of partial sums (S.O.P.S.) :

$$\sum u_n$$

Define a seq. $\langle s_n \rangle$ as:

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$s_3 = u_1 + u_2 + u_3$$

$$\vdots \qquad\qquad \vdots$$

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

Ex: Show that the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots-$$

is convergent.

$$\rightarrow \sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}$$

Let $\langle s_n\rangle$ be the S.O.P.S.

$$s_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}$$

$$=\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\cdots+\left(\cancel{\tfrac{1}{n}}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$$\lim s_n=\lim\left(1-\frac{1}{n+1}\right)=1-0=1$$

$$\therefore \langle s_n\rangle \rightarrow 1.$$

Let $\langle s_n \rangle$ be the s.o.p.s.

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

$\vdots \qquad \vdots \qquad \vdots$

$\langle 1, 0, 1, 0, --- \rangle$

oscillates finitely.

$\therefore \sum u_n$ is cgt.

i.e. $\sum \frac{1}{n\cdot(n+1)}$ is cgt

Sum = 1.

Ex: Show that the series $\sum (-1)^{n-1}$ oscillates.

$\rightarrow \sum u_n, \quad u_n = (-1)^{n-1}$

is also cgt.

where $s_n = u_1 + u_2 + \cdots + u_n$

let $\lim s_n = l$.

Now $s_n - s_{n-1}$

$= (u_1 + u_2 + \cdots + u_n)$

$- (u_1 + u_2 + \cdots + u_{n-1})$

$s_n - s_{n-1} = u_n$

$\Rightarrow \lim (s_n - s_{n-1}) = \lim u_n$

$\therefore \sum u_n$ also oscillates.

A necessary condition for Convergence:

Stat: If $\sum u_n$ converges then $\lim u_n = 0$

proof: Suppose $\sum u_n$ is cgt.

$\therefore$ its S.O.P.S. $\langle s_n \rangle$

Consider the example

$$\sum \frac{1}{n}$$

$$\lim \frac{1}{n} = 0$$

But $\sum \frac{1}{n}$ is not cgt.

(2) $\lim u_n \neq 0 \Rightarrow \sum u_n$ is not cgt.

$$\Rightarrow \lim s_n - \lim s_{n-1} = \lim u_n$$

$$\Rightarrow l - l = \lim u_n$$

$$\Rightarrow \lim u_n = 0$$

Note: (1) $\lim u_n = 0$ is not a sufficient condition for the convergence of $\sum u_n$.

Cauchy's principle of convergence

Stat: A necessary and sufficient condition for a series $\sum u_n$ to converge is that for each $\varepsilon > 0$ $\exists$ a +ve integer m s.t.

$$|u_{m+1} + u_{m+2} + \cdots + u_n| < \varepsilon \quad \forall n \geq m$$

or

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon$$

$\forall n \geq m$ & $p \geq 1$.

Ex: $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

$\rightarrow$ $u_n = \frac{n}{n+1}$

$$\lim u_n = \lim \frac{n}{n+1}$$

$$= \lim \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{1+0} = 1 \neq 0$$

$\therefore$ The given series is not cgt.

i.e. $\Leftrightarrow |u_{m+1}+u_{m+2}+\cdots+u_n|<\varepsilon$

$\forall n \geq m$.

Prob: Show that the series $\sum \frac{1}{n}$ does not converge.

→ Let, if possible, $\sum \frac{1}{n}$ is cgt.

∴ its S.O.P.S. $\langle s_n \rangle$ will also converge.

Let $\langle s_n \rangle$ be the S.O.P.S of the series $\sum u_n$

$\sum u_n$ is cgt $\Leftrightarrow \langle s_n \rangle$ is cgt.

$\Leftrightarrow$ for each $\varepsilon>0$, $\exists$ a +ve int. m s.t.

$$|s_n - s_m| < \varepsilon \quad \forall n \geq m$$

i.e. $\Leftrightarrow |(u_1+u_2+\cdots+u_n) - (u_1+u_2+\cdots+u_m)| < \varepsilon$, $\forall n \geq m$

$$= \left| 1 + \frac{1}{2} + \cdots \cdots + \frac{1}{2m} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) \right|$$

$$= \left| \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \cdots \text{ m times}$$

$$= m \cdot \frac{1}{2m} = \frac{1}{2}$$

This is a contra. to (1)

$\therefore$ By Cauchy's principle of convergence, $\langle s_n \rangle$ is a Cauchy seq.

$\therefore$ Corr. to $\varepsilon = \frac{1}{2}$, $\exists$ a +ve int. m s.t.

$$|s_n - s_m| < \frac{1}{2} \quad \forall n \geq m \qquad (1)$$

Let $n = 2m > m$

$$|s_{2m} - s_m|$$

(HW) of last class

$$a_n = \sqrt{a_{n-1} \cdot a_{n-2}} \quad \forall n > 2$$

$\rightarrow \quad a_1 = a_2$

$$a_3 = \sqrt{a_2 \cdot a_1} = \sqrt{a_1^2} = a_1$$

$$a_4 = a_1$$

$\vdots$

$$a_n = a_1 \quad \forall n.$$

$\therefore \lim a_n = a_1$

$\therefore$ our supposition is wrong.

$\therefore \{\frac{1}{n}\}$ is not cgt.

& so on

$a_1 < a_3 < a_5 < \cdots < a_4 < a_2$

$\langle a_{2n-1} \rangle \longrightarrow \uparrow$ bdd above by a_2

$\therefore$ it is cgt.

$\langle a_{2n} \rangle \longrightarrow \downarrow$ bdd below by a_1

$\therefore$ it is cgt.

Let $a_1 < a_2$

$a_3 = \sqrt{a_1 a_2}$

$\therefore$ a_3 is the G.M. of a_1 & a_2

$\therefore$ $a_1 < a_3 < a_2$

$a_4 = \sqrt{a_3 a_2}$

$\Rightarrow$ a_4 is the G.M of a_2 & a_3

$\therefore$ $a_3 < a_4 < a_2$

& so on.

$$a_n^2 \cdot a_{n-1} = a_2^2 \cdot a_1$$

$$\lim\left(a_n^2 \cdot a_{n-1}\right) = a_2^2 \cdot a_1$$

$$\Rightarrow l^2 \cdot l = a_2^2 \cdot a_1$$

$$\Rightarrow l^3 = a_2^2 \cdot a_1$$

$$\Rightarrow \underline{l = \left(a_2^2 \cdot a_1\right)^{\frac{1}{3}}}$$

$\|^{ly}$, we can consider the case $a_1 > a_2$ and will have same limit.

$$\therefore a_n^2 = a_{n-1} \cdot a_{n-2}$$

$\underline{n=3}$

$$a_3^2 = a_2 \cdot a_1$$

$$\Rightarrow a_3^2 \cdot a_2 = a_2^2 \cdot a_1$$

$\underline{n=4}$

$$a_4^2 = a_3 \cdot a_2$$

$$\Rightarrow \underline{a_4^2 \cdot a_3} = a_3^2 \cdot a_2$$

$$= a_2^2 \cdot a_1$$

$l'(l'-l)=0$

$\Rightarrow l=l' \quad [\because l'\neq 0]$

$\therefore \langle a_{2n}\rangle$ and $\langle a_{2n-1}\rangle$ Converge to l.

$\therefore \lim a_n = l$

Let $\lim a_{2n-1}=l$ and $\lim a_{2n}=l'$

we have

$$a_{2n}=\sqrt{a_{2n-1}\cdot a_{2n-2}}$$

$$\Rightarrow (a_{2n})^2 = a_{2n-1}\, a_{2n-2}$$

$$\Rightarrow \lim (a_{2n})^2 = \lim (a_{2n-1}\cdot a_{2n-2})$$

$$\Rightarrow (l')^2 = l\cdot l'$$

$$\Rightarrow (l')^2 - l\,l' = 0$$