KGS IAS

Maths Optional

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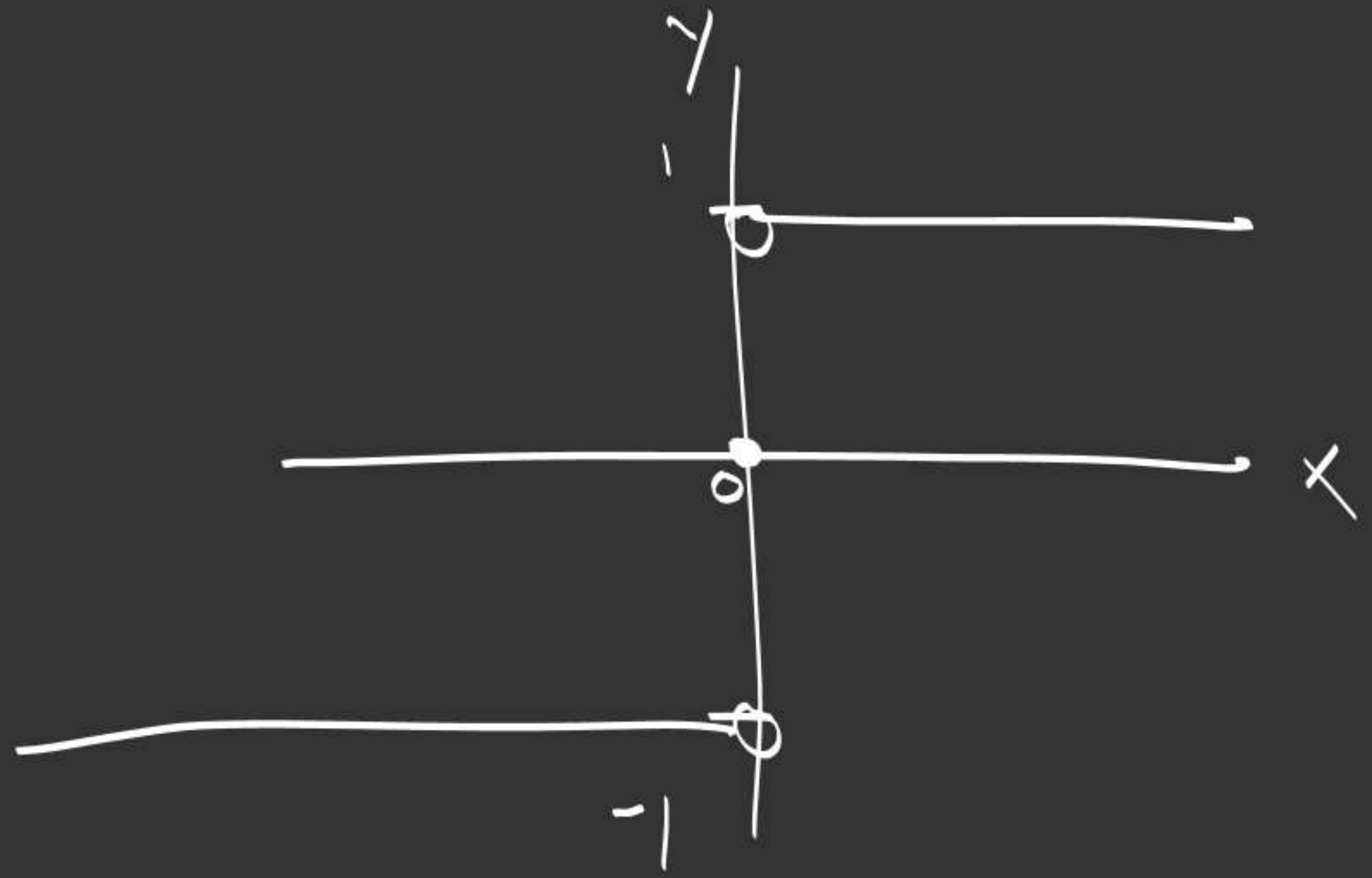


Domain $= \mathbb{R}$

Range $= \{-1, 0, 1\}$.

Signum function

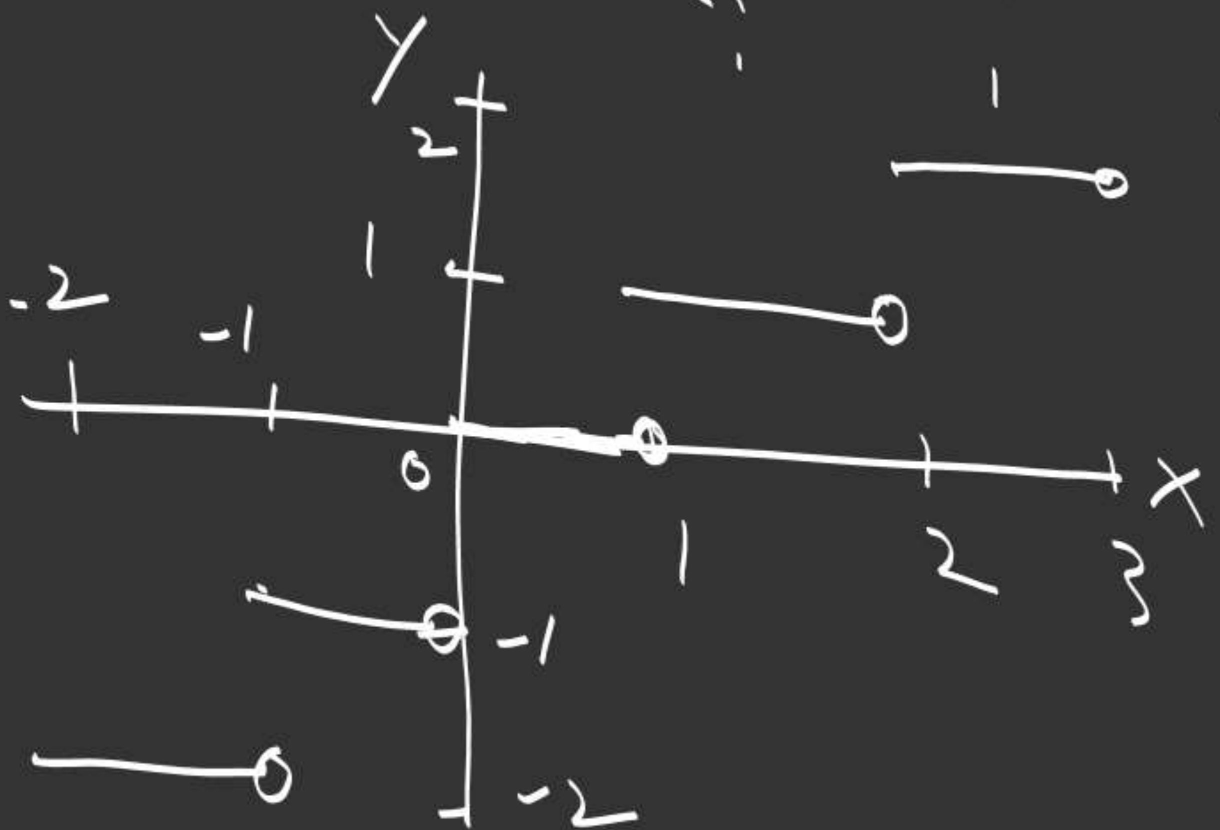
$f : \mathbb{R} \to \mathbb{R}$

defined by

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$= \begin{cases} \frac{|x|}{x}, & x \neq 0 \; \left(x > 0, \; x < 0\right) \\ 0, & x = 0 \end{cases}$$

$$f(x) = [x] = \begin{cases} \vdots & \vdots \\ -2 & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ 3 & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



Greatest integer fn (step function or Integral value fn.)

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = [x]$ = integral part of x [greatest integer $\le x$].

Domain $= \mathbb{R}$

Range $= \mathbb{Z}$ (set of int.)

Ex: $x = 3$

$3 \le x < 4$

$[x] = 3$

Exponential fn:

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = a^x$, $a \neq 0, 1$, $a > 0$

Domain $= \mathbb{R}$

Range $=]0, \infty[$

Note: $x \in \mathbb{R}$, $\exists$ a unique int. n s.t.

$n \le x < n+1$

$\therefore [x] = n$.

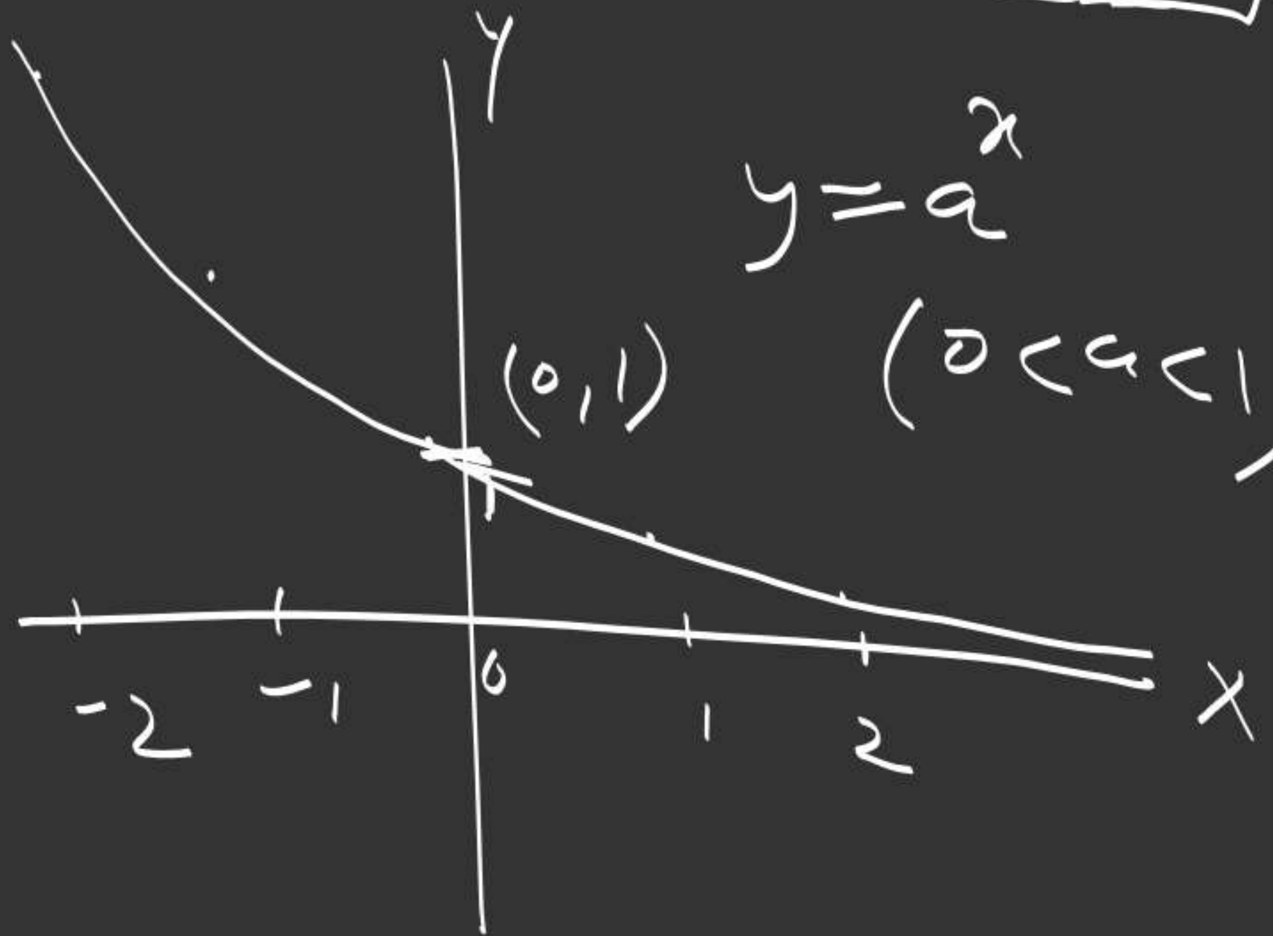
Ex: $x = 2.5$

$2 \le 2.5 < 3$

$[2.5] = 2$

$y = \left(\frac{1}{2}\right)^x$ $\quad$ $0 < a < 1$

x	0	1	2	-1	-2
y	1	$\frac{1}{2}$	$\frac{1}{4}$	2	4



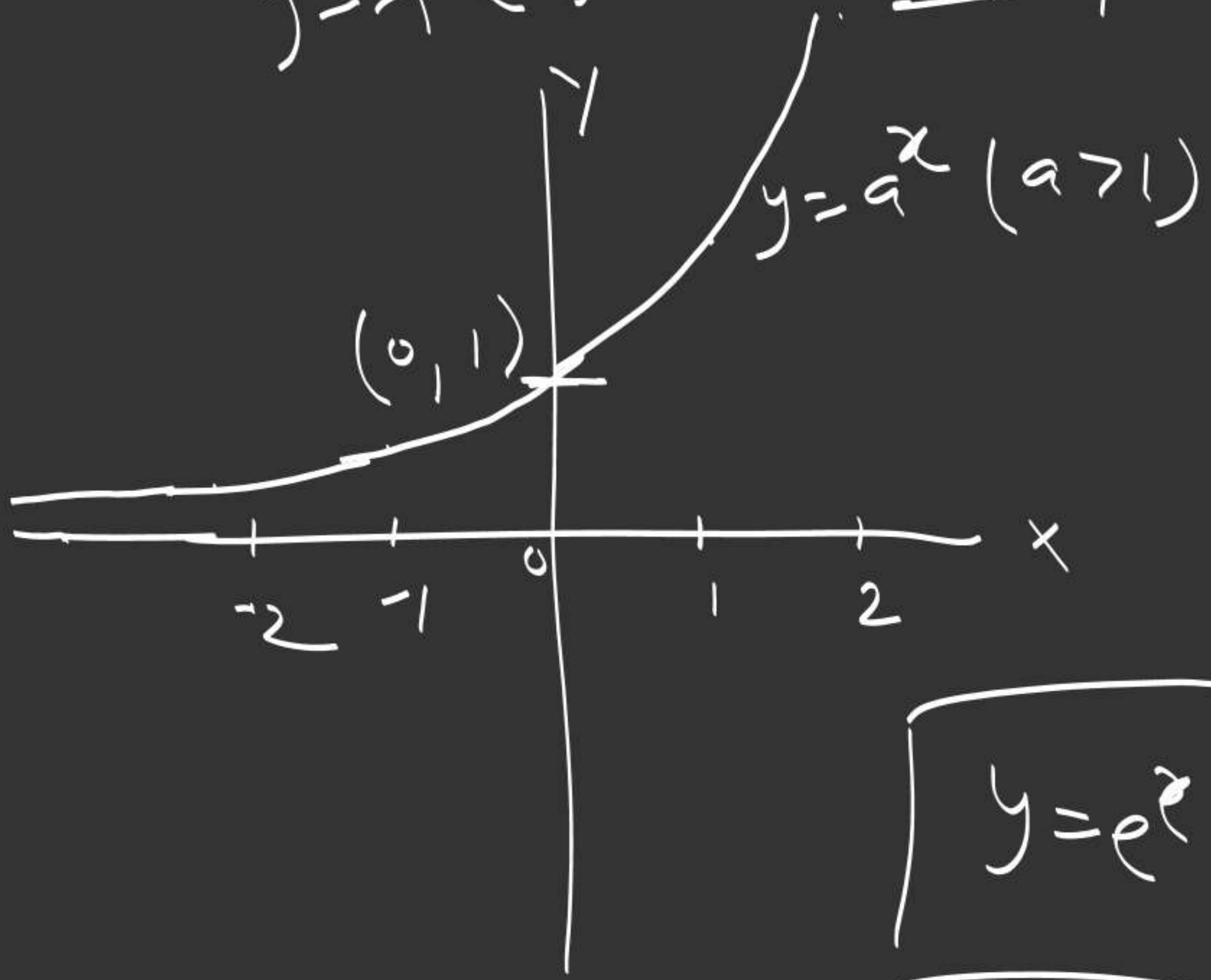
$y = a^x$

$(0 < a < 1)$

$a = 2 \ (>1)$

$y = f(x) = 2^x$

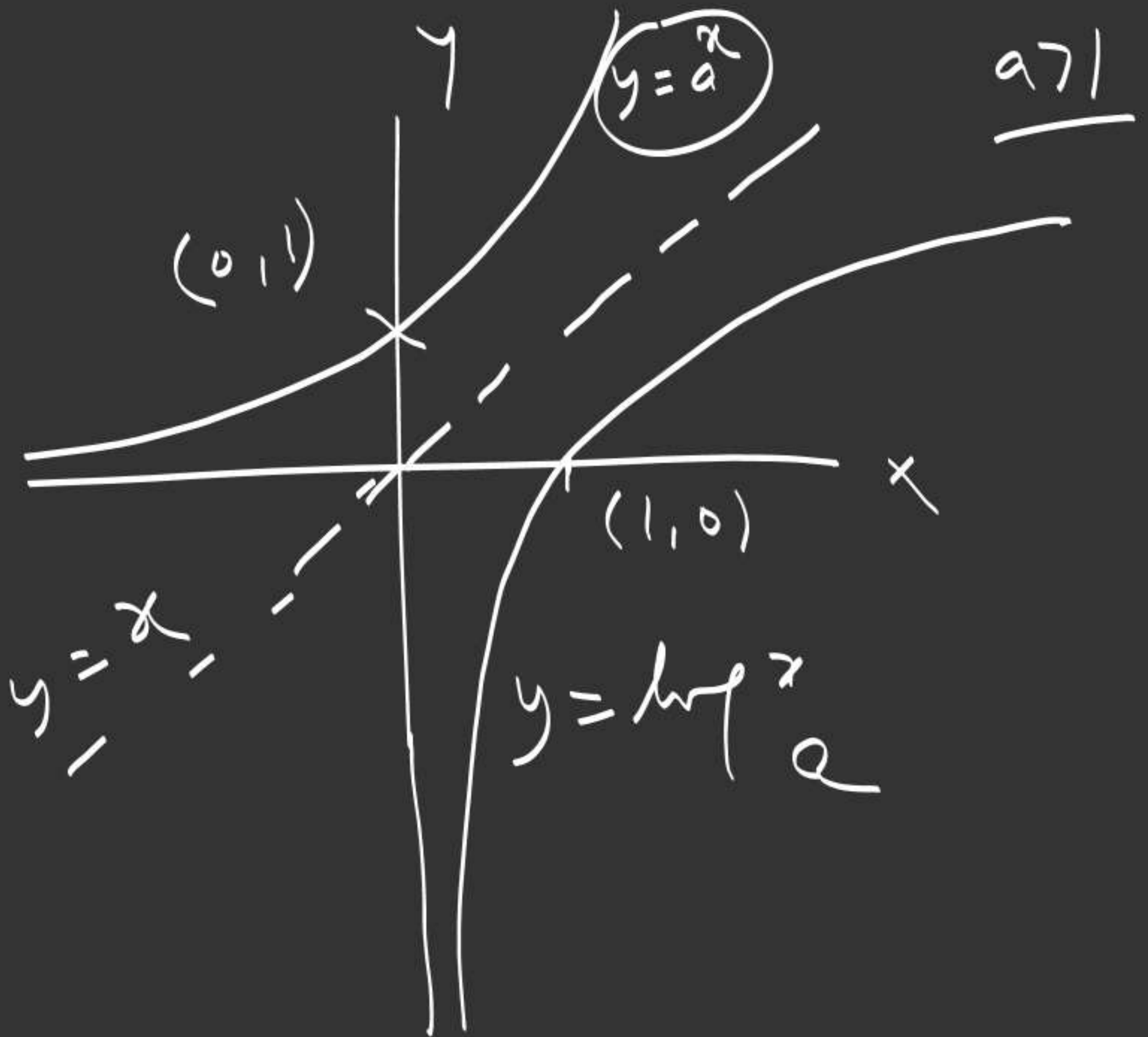
x	0	2	-1	-2
y	1	4	$\frac{1}{2}$	$\frac{1}{4}$



$$\boxed{y = e^x}$$

$e > 1$

$y = a^x$ $(a>1)$ $\qquad$ $y = \log_a x$



Logarithmic fn:

$$f(x) = \log_a x$$

$x>0$

$a>0,\ a\neq 1$

$f: \mathbb{R}^+ \to \mathbb{R}$

Domain $=]0, \infty[$

Range $= \mathbb{R}$.

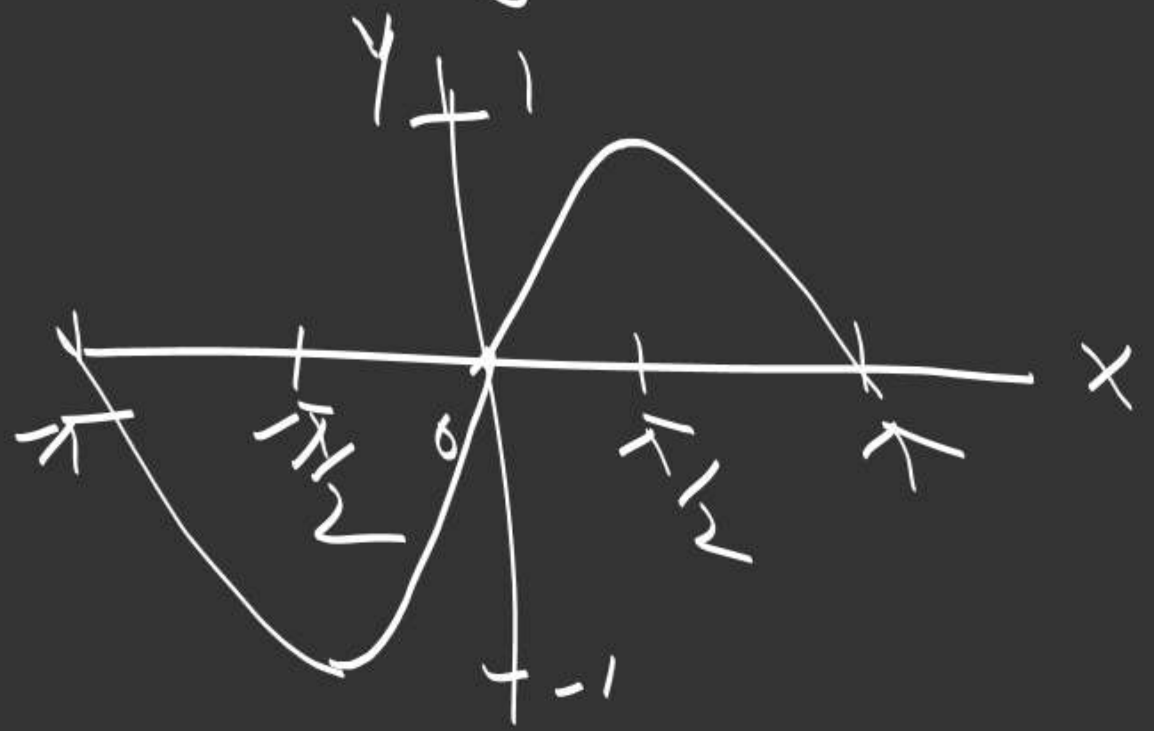
Trigonometric fn.

① $y = f(x) = \sin x$.

$f : \mathbb{R} \to \mathbb{R}$

Domain $= \mathbb{R}$

Range $= [-1, 1]$

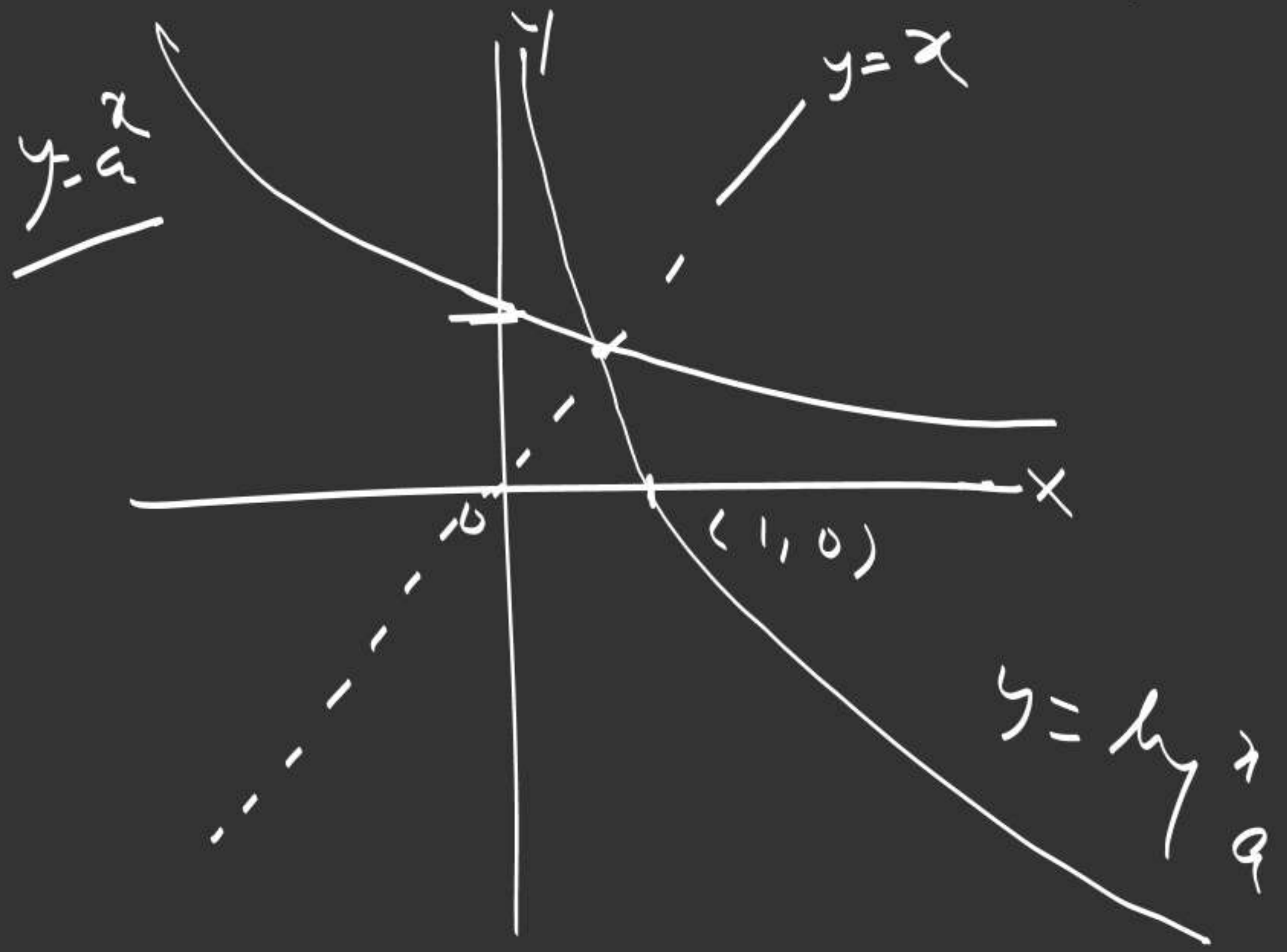


→ $y = a^x$

$0 < a < 1$

$y = \log_a x$

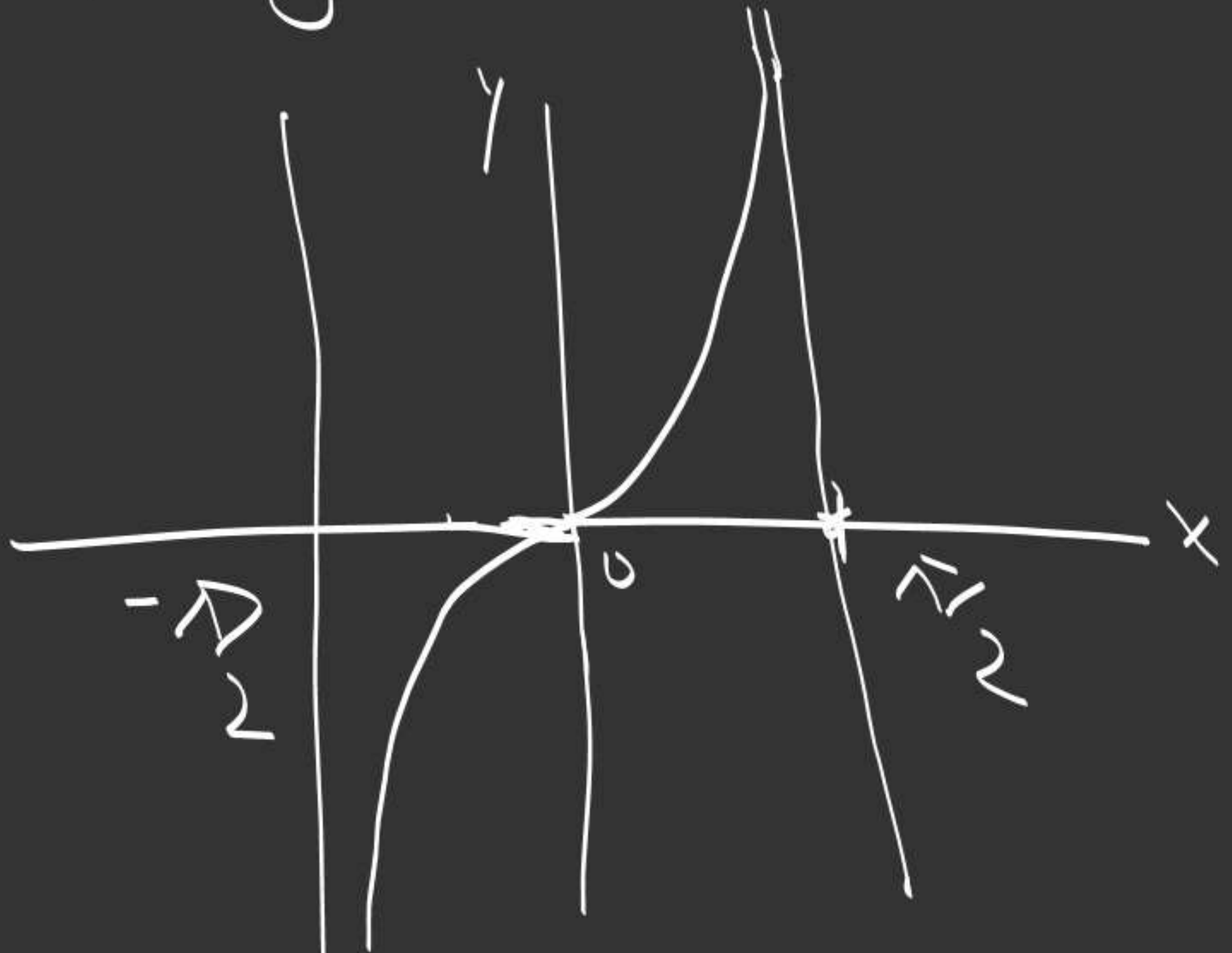
$0 < a < 1$



(iii) $y = f(x) = \tan x = \dfrac{\sin x}{\cos x}$

Domain $= \mathbb{R} \sim \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\}$

Range $= \mathbb{R}$

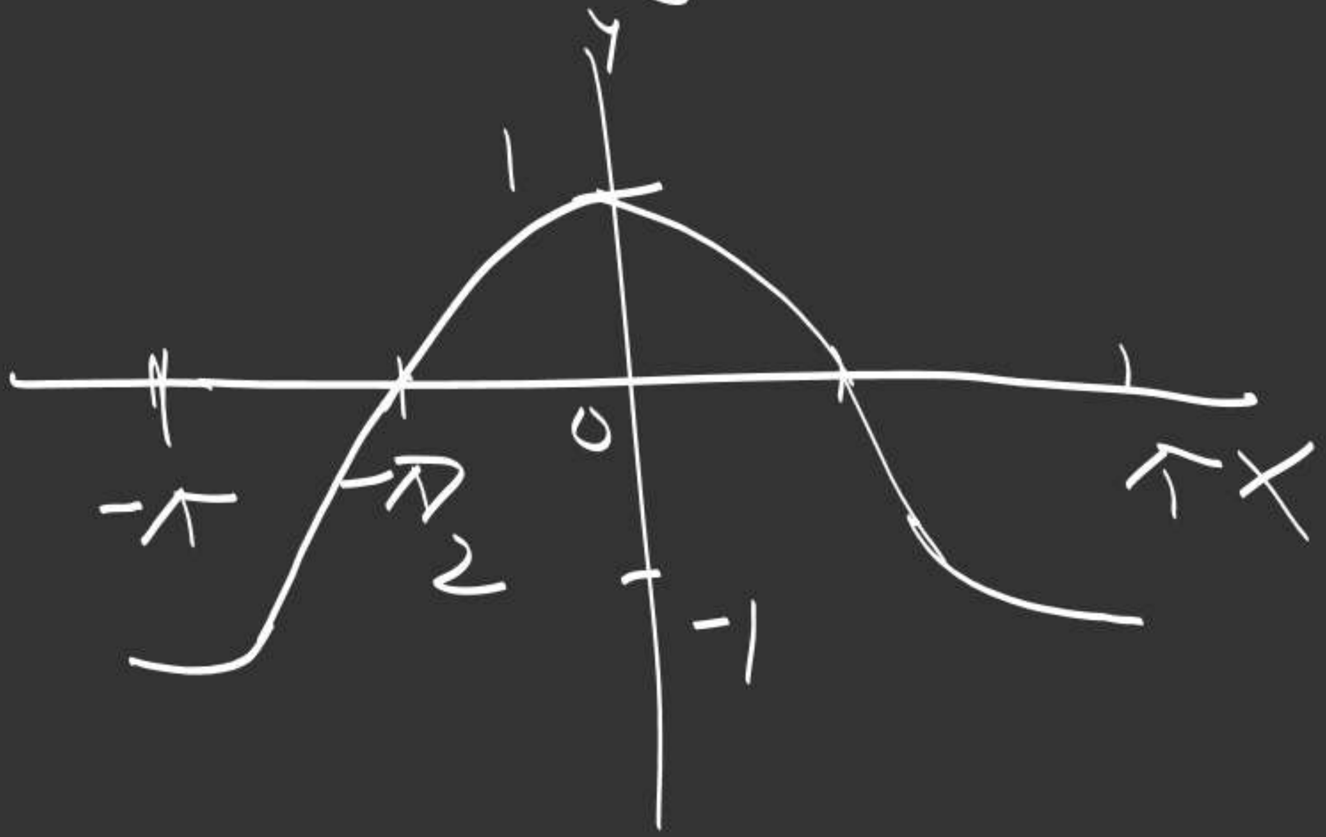


(ii) $y = f(x) = \cos x$

$f : \mathbb{R} \to \mathbb{R}$

Domain $= \mathbb{R}$

Range $= [-1, 1]$

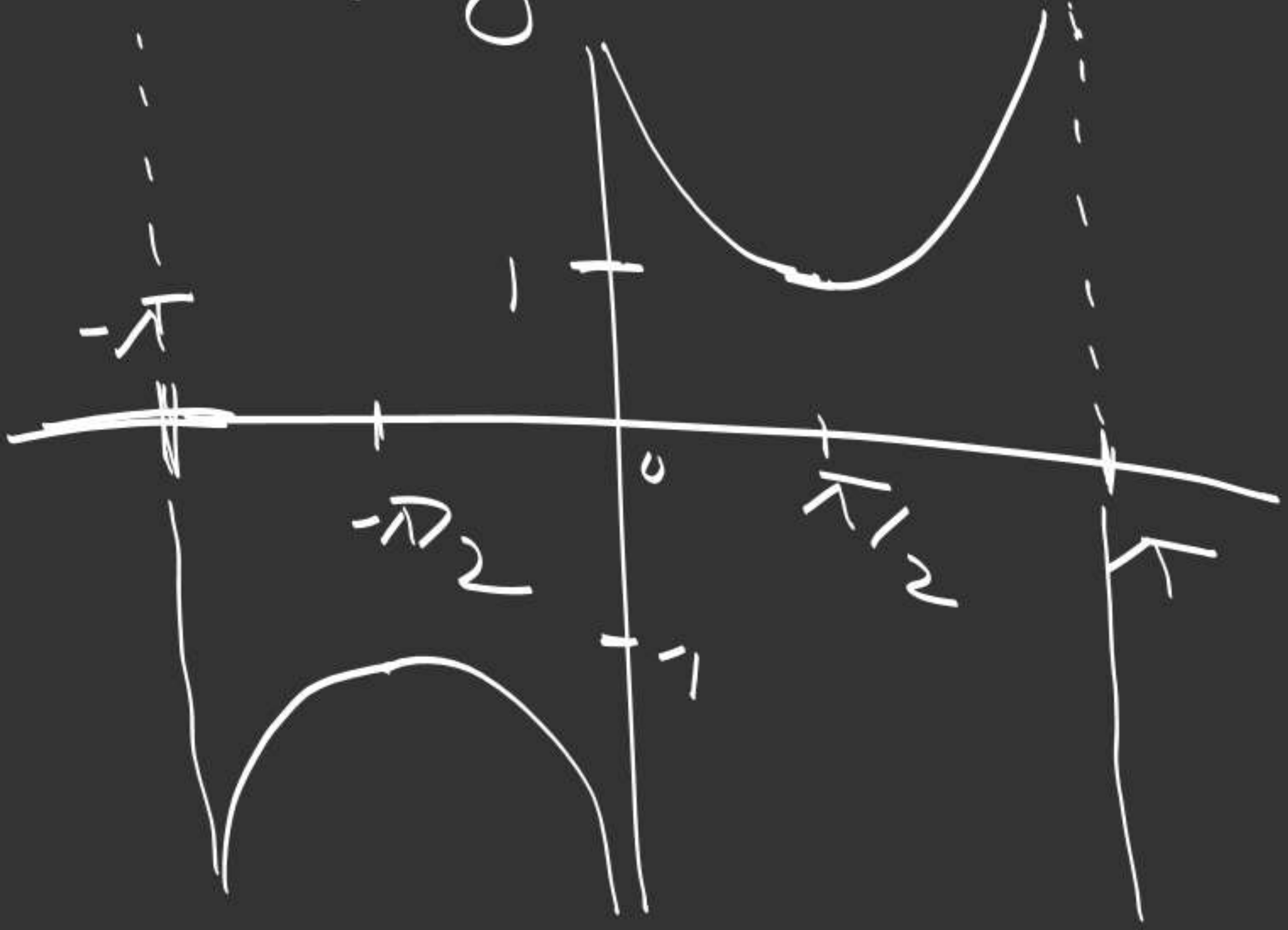


(V) $y = f(x) = \operatorname{cosec} x$

$= \dfrac{1}{\sin x}$

Domain $= \mathbb{R} \sim \{n\pi : n \in \mathbb{Z}\}$

Range $= \mathbb{R} \sim]-1, 1[$



(VI) $y = f(x) = \cot x = \dfrac{\cos x}{\sin x}$

Domain $= \mathbb{R} \sim \{n\pi : n \in \mathbb{Z}\}$

Range $= \mathbb{R}$.

Boundedness of a function

A function f is said to be bounded if its range is bdd. Otherwise unbounded.

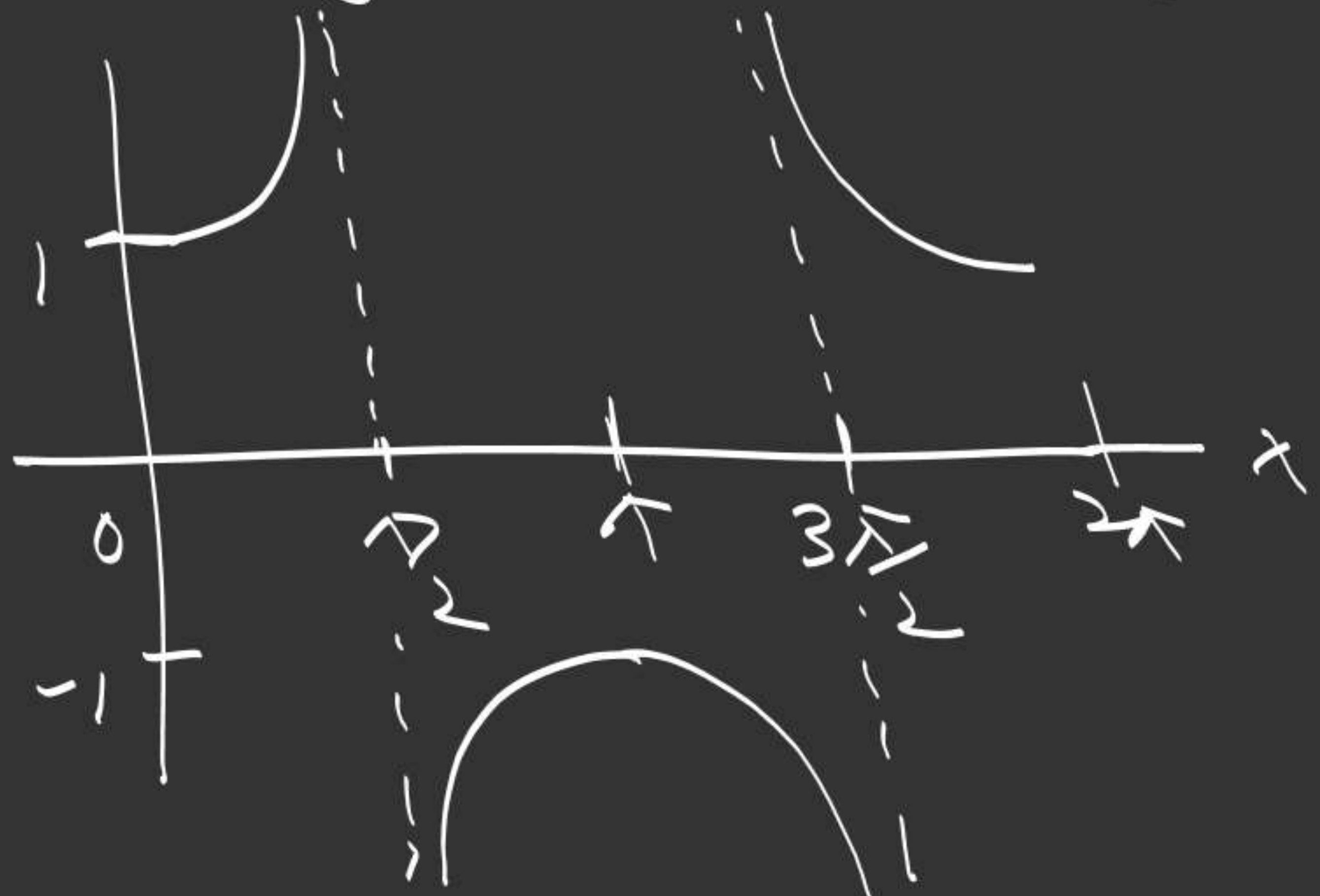
or

A function f is said to be bdd on a domain D if $\exists$ real numbers k and K

(vi) $y = f(x) = \sec x = \dfrac{1}{\cos x}$

Domain $= \mathbb{R} \setminus \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\}$

Range $= \mathbb{R} \setminus]-1, 1[$



Ex: $f(x) = \sin x$ is bdd.

$$-1 \le f(x) \le 1 \quad \forall x \in \mathbb{R} = \text{Dom.}$$

Ex: $f(x) = \tan x$ is unbdd.

$$\text{Range} =]-\infty, \infty[$$

Range is not bdd.

s.t

$$k \le f(x) \le K \quad \forall x \in D.$$

or

A function f is said to bdd on a Domain D.

if $\exists$ a +ve real no M

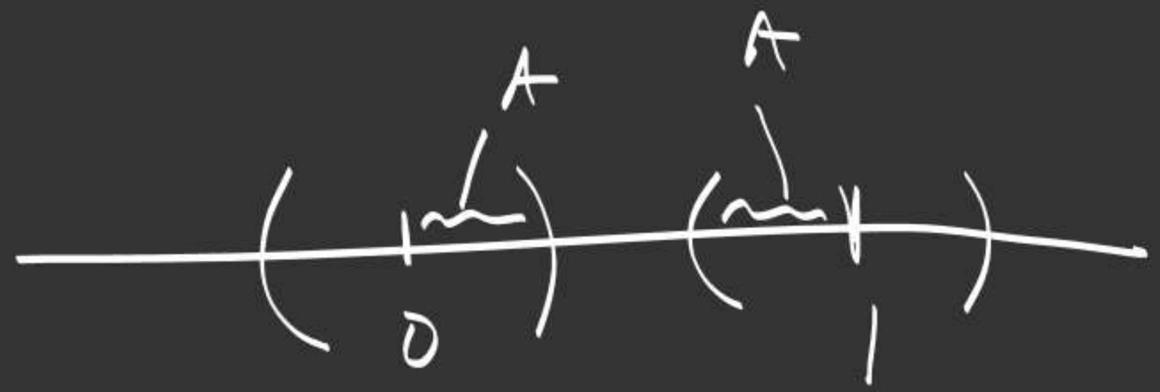
s.t

$$|f(x)| \le M \quad \forall x \in D$$

$\iff |x-c|<\delta \,,\ x\neq c$

$\iff 0<|x-c|<\delta$ (As $x\neq c$)

<u>Ex</u> $A =]0,1[$



set of limit ~~pts~~ of $A = [0,1]$

$A\subseteq\mathbb{R}$, <u>c</u>

c is a limit point of A, if for each $\delta>0$

$\exists\, x\in A$ $(x\neq c)$

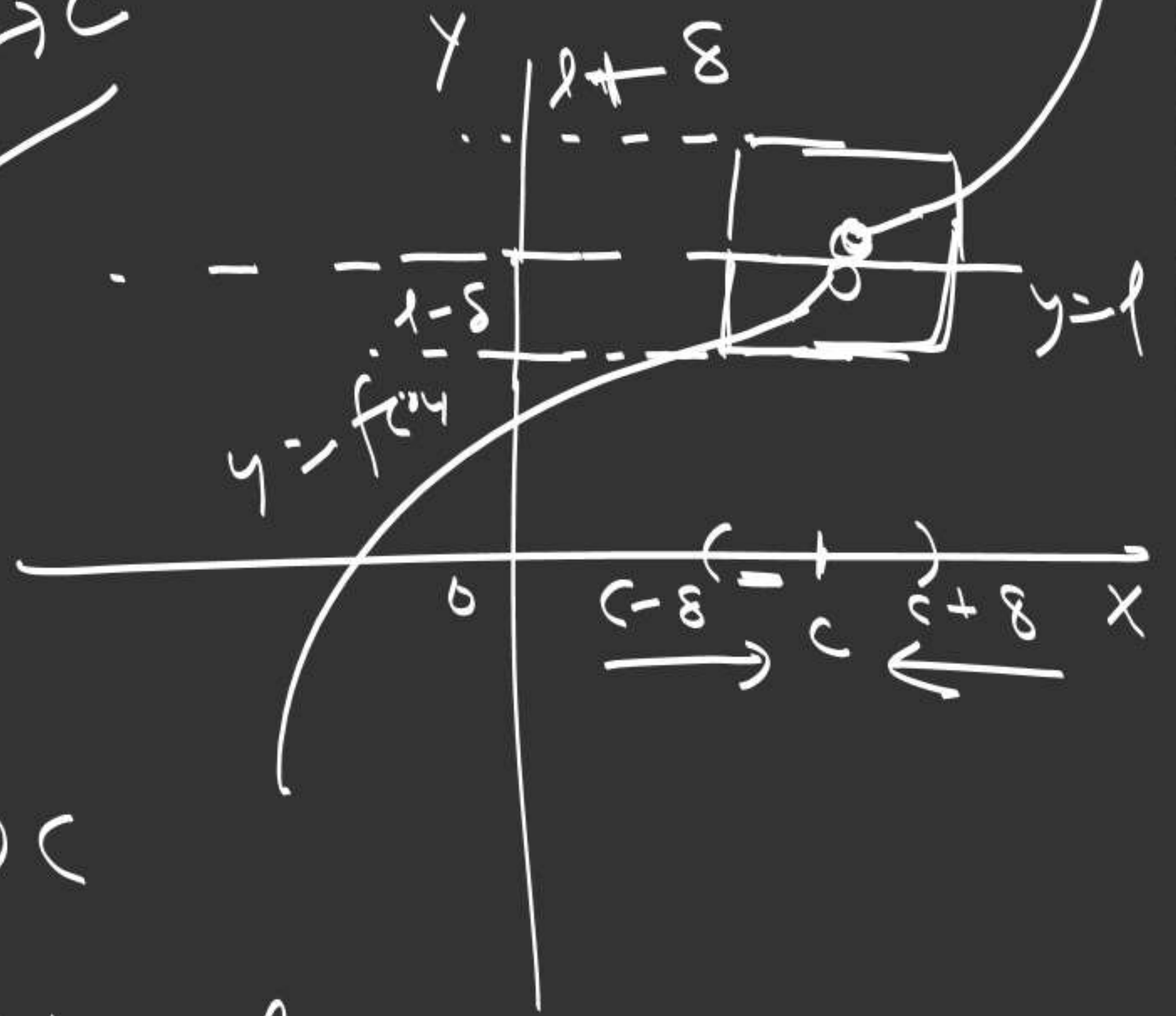
s.t. <u>$x\in]c-\delta, c+\delta[$</u>

$\iff c-\delta<x<c+\delta \,,\ x\neq c$

$\iff -\delta<x-c<\delta \,,\ x\neq c$

Limit of a function

$$\lim_{x \to c} f(x) = l$$



As $x \to c$

$f(x) \to l$.

Let f be a function defined on some nbhd of c except possibly at c. f is said to approach a real number l as x approaches c if for each $\varepsilon > 0$, $\exists$ some $\delta > 0$ (depending on ε) s.t.

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - c| < \delta$$

i.e. $l - \varepsilon < f(x) < l + \varepsilon$ when $c - \delta < x < c + \delta$, $x \neq c$.

Ex: using $\varepsilon-\delta$ definition to prove that

$$\lim_{x \to 0} \frac{x^2}{|x|} = 0.$$

$\rightarrow$ Let $\varepsilon > 0$ be given.

$$\left| \frac{x^2}{|x|} - 0 \right| = \left| \frac{x^2}{|x|} \right| = \frac{|x|^2}{|x|} = |x|$$

So. $\left| \frac{x^2}{|x|} - 0 \right| < \varepsilon$ whenever $0 < |x| < \delta = \varepsilon$

2nd defn: Let $A \subseteq \mathbb{R}$ and c is a limit point of A. For a function $f : A \to \mathbb{R}$, f is said to approach a real number l as x approaches c if for each $\varepsilon > 0$, $\exists$ some $\delta > 0$ (depending on ε) s.t.

$$|f(x) - l| < \varepsilon \text{ whenever } x \in A, \; 0 < |x - c| < \delta.$$

→ Let $\varepsilon > 0$ be given.

$$|x \sin \tfrac{1}{x} - 0| = |x \sin \tfrac{1}{x}|$$
$$= |x| \cdot |\sin \tfrac{1}{x}|$$
$$\leq |x|$$

[As $|\sin x| \leq 1$]

$\therefore$ $|x \sin \tfrac{1}{x} - 0| < \varepsilon$ whenever $0 < |x - 0| < \delta = \varepsilon$

Choosing $\delta = \varepsilon$,

$\therefore$ For any $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

Choose $\delta = \varepsilon$,

$\therefore$ For any $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$$\left| \frac{x^2}{|x|} - 0 \right| < \varepsilon \text{ whenever } 0 < |x - 0| < \delta$$

$$\therefore \lim \frac{x^2}{|x|} = 0$$

Ex: By ε-δ definition, Prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

$f(x)$ diverges as $x \to c$.

(III) Limit is unique.

(IV) In order to prove that $\lim_{x \to c} f(x) \neq l$, it is sufficient to find some $\varepsilon > 0$ s.t. for each $\delta > 0$ we can find some x s.t. $0 < |x - c| < \delta$ but $|f(x) - l| \geq \varepsilon$

$|x \sin 1/x - 0| < \varepsilon$ whenever

$0 < |x - 0| < \delta$

$\therefore \lim x \sin 1/x = 0$

<u>Note</u>: ① $\lim_{x \to c} f(x) = l$

As $x \to c$, $f(x)$ converge to l.

(ii) $\lim_{x \to c} f(x)$ doesn't exist.