KGS IAS

Maths Optional

By Dhrub Singh Sir

exist.

(iii) Disc. of 2nd kind from right at 'c' if $\lim_{x \to c+} f(x)$ does not exist.

<u>Ex:</u>

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x \to 0-} \sin\frac{1}{x}$ doesn't exist.

Discontinuity of 2nd kind

(i) Disc. of 2nd kind at 'c' if

Neither $\lim_{x \to c-} f(x)$ nor $\lim_{x \to c+} f(x)$ exists.

(ii) Disc. of 2nd kind from left at 'c' if $\lim_{x \to c-} f(x)$ does not or

(ii) $\lim_{x \to c+} f(x)$ does not exist.

$\lim_{x \to c-} f(x)$ exists and may or may not be equal to $f(c)$.

Ex: $f(x) = \begin{cases} 1-x & \text{if } x > 0 \\ \sin\frac{1}{x} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\lim_{x \to 0+} \sin\frac{1}{x}$ does not exist.

Dis. of 2nd kind at '0'.

Mixed Disc:

(i) $\lim_{x \to c-} f(x)$ does not exist.

$\lim_{x \to c+} f(x)$ exists and may or may not be equal to $f(c)$.

$\rightarrow \lim_{x \to 2^+} f(x) = \lim_{\substack{x \to 2 \\ x > 2}} \frac{1}{x-2}$

$= +\infty.$

$\lim_{x \to 2^-} f(x) = \lim_{\substack{x \to 2 \\ x < 2}} \frac{1}{x-2}$

$= -\infty.$

f has infinite disc. at $x = 2$.

Infinite Discont.

one or both of

$\lim_{x \to c^-} f(x)$ & $\lim_{x \to c^+} f(x)$

is $+\infty$ $(-\infty)$.

Ex:

$$f(x) = \begin{cases} \frac{1}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

(ii) $\lim_{x \to c} (f \cdot g)(x) = (f \cdot g)(c)$

$\therefore$ $f \cdot g$ is cts at $x = c$.

(iii) $\lim_{x \to c} \frac{f}{g}(x) = \frac{f}{g}(c)$

$g(c) \neq 0$

$g(x) \neq 0$

$\therefore$ $\frac{f}{g}$ is also cts at $x = c$.

Algebra of cts fn:

Let f & g are cts at $x = c$.

$\therefore$ $\lim_{x \to c} f(x) = f(c)$, $\lim_{x \to c} g(x) = g(c)$.

(i) $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} [f(x) \pm g(x)]$

$= \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

$= f(c) \pm g(c)$

$= (f \pm g)(c)$

$\therefore$ $f + g$ is cts at $x = c$.

→ Let $c \in \mathbb{R}$ be any real no.

Let $\langle x_n \rangle \in \mathbb{R}$ converging to c.

$$f(x_n) = K \quad \forall n \in \mathbb{N}$$

$$f(c) = K$$

$$\langle f(x_n) \rangle \equiv \langle K, K, \dots \rangle \to K = f(c)$$

$\therefore f$ is cts at $x = c$

(iv) $\lim_{x \to c} (Kf)(x) = (Kf)(c)$

$\underline{K \in \mathbb{R}}$

$\therefore Kf$ is also cts at $x = c$

Prob: Show that the constant function $f(x) = K$ is cts in $\mathbb{R}$.

Converging to c.

Consider $\langle f(x_n) \rangle$

$$f(x_n) = x_n \quad \forall n \in \mathbb{N}$$

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n$$

$$= c$$

$$= f(c)$$

$\therefore \langle f(x_n) \rangle \to f(c)$

$\therefore f$ is cts at $x = c$.

$\therefore c$ is any point of $\mathbb{R}$.

$\therefore f$ is cts in $\mathbb{R}$

<u>prob:</u> $f(x) = x$ is cts in $\mathbb{R}$.

$\to$ Let c be a point in $\mathbb{R}$.

$$f(c) = c$$

Let $\langle x_n \rangle$ be a seq. in $\mathbb{R}$

$f(x_n) = x_n^2 \quad \forall n \in \mathbb{N}.$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2$$

$$= \lim_{n\to\infty} x_n \times \lim_{n\to\infty} x_n$$

$$= c \times c = c^2$$

$$= f(c)$$

$\therefore \langle f(x_n) \rangle \to f(c)$

$\therefore$ f is cts at $x = c$

$\because$ c is any pt. of $\mathbb{R}$

c is any point of $\mathbb{R}$.

$\therefore$ f is cts in $\mathbb{R}$.

<u>prob</u>: $f(x) = x^2$ is cts in $\mathbb{R}$.

$\rightarrow$ Let c be any real no.

$$f(c) = c^2$$

Let $\langle x_n \rangle$ be an seq. in $\mathbb{R}$ converging to 'c'.

or

$$\lim_{x \to 0-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0+} \frac{1}{x} = +\infty.$$

$\therefore f$ is cts in $\mathbb{R}$.

<u>prob:</u> Show that $f(x) = \frac{1}{x}$ is not cts at $x = 0$.

$\longrightarrow$ f is not defined at $\underline{x = 0}$.

or

$x_n = \frac{1}{n}$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

$\therefore$ f is discts at $x=0$.

HW

prob

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Check cont. at $x=0$

Ans: Discont.

prob

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Check cont. at $x=0$.

$\longrightarrow$ $\lim_{x\to 0^-} f(x) = -1$

$\lim_{x\to 0^+} f(x) = 1$

$f(0) = 0$

$$\lim_{x\to 0+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$$

$$= \lim_{x\to 0+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = \frac{1-0}{1+0} = 1$$

$$f(0) = 1$$

$$\lim_{x\to 0-} f(x) \neq \lim_{x\to 0+} f(x)$$

$\therefore$ f is discts at $x=0$.

prob:

$$f(x) = \begin{cases} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

check cont. at $x=0$.

$$\rightarrow \lim_{x\to 0-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$$

$$= \lim_{x\to 0-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = \frac{0-1}{0+1} = -1.$$

Prob

$$f(x) = \begin{cases} (x-a)\cdot \dfrac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}, & x \neq a \\ 0, & x = a \end{cases}$$

Check cont. at $x = a$.

$\longrightarrow \lim\limits_{x \to a-} \dfrac{1}{x-a} = -\infty$

$\lim\limits_{x \to a+} \dfrac{1}{x-a} = +\infty$

$$\lim_{x \to a-} (x-a)\cdot \frac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}$$

$$= 0 \times \frac{0-1}{0+1} = 0$$

$$\lim_{x \to a+} (x-a) \frac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}$$

$$= \lim_{x \to a+} (x-a) \frac{1 - e^{-\frac{1}{(x-a)}}}{1 + e^{-\frac{1}{(x-a)}}}$$

$$= 0 \times \frac{1-0}{1+0} = 0$$

HW

Prob: Discuss the continuity of the following functions at $x=0$.

(I) $f(x)=\begin{cases} 2^{1/x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

(II) $f(x)=\begin{cases} x\cos\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0. \end{cases}$

Ans (I) Disc.
(II) cts.

$f(a)=0$

$\therefore f$ is cts at $x=a$.

HW

Prob:

$f(x)=\begin{cases} \dfrac{x\cdot e^{1/x}}{1+e^{1/x}} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

Check cont. at $x=0$.

Ans: cont.

$$f(x)=\begin{cases}-2x+1, & x<0\\ 1, & 0\le x<1\\ 2x-1, & x\geq 1\end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0\\ x<0}} f(x) = \lim_{\substack{x\to 0\\ x<0}} -2x+1 = 1$$

prob: check the continuity of $f(x)=|x|+|x-1|$ at $x=0,1$.

$$f(x)=\begin{cases}-x-(x-1), & x<0\\ x-(x-1), & 0<x<1\\ x+x-1 & x>1\end{cases}$$

Prob:

$$f(x) = \begin{cases} \dfrac{x-|x|}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Check continuity at $x = 0$.

$\rightarrow$ $\displaystyle\lim_{x \to 0-} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} \frac{x - |x|}{x}$

$\displaystyle = \lim_{\substack{x \to 0 \\ x < 0}} \frac{x + x}{x} = 2$

$\displaystyle\lim_{x \to 0+} f(x)$

$\displaystyle = \lim_{\substack{x \to 0 \\ x > 0}} 1 = 1$

$f(0) = 1$

$\therefore$ f is cts at $x = 0$

f is cts at $x = 1$ (check it!)

prob: Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ c, & x = 0 \\ \dfrac{(x+bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{b \cdot x^{\frac{3}{2}}}, & x > 0 \end{cases}$$

Determine the values of a, b, c for which the fn. is cts at $x = 0$.

$$\lim_{x \to 0+} f(x)$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{x - |x|}{x}$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{x - x}{x} = 0$$

$$f(0) = 1$$

So, f is disc. at $x = 0$

$$\lim_{x\to 0+} f(x)$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{(x+bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{b\cdot x^{\frac{3}{2}}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{x^{\frac{1}{2}}(1+bx)^{\frac{1}{2}} - x^{\frac{1}{2}}}{b\cdot x^{\frac{3}{2}}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\sqrt{1+bx} - 1}{b\cdot x}$$

$\because$ f is cts at $x=0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$\lim_{x\to 0-} f(x) = \lim_{\substack{x\to 0\\ x<0}} \frac{\sin(a+1)\cdot x + \sin x}{x}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \left[(a+1)\times\frac{\sin(a+1)\cdot x}{(a+1)\cdot x} + \frac{\sin x}{x} \right]$$

$$= (a+1)\times 1 + 1 = a+2$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$a+2 = \frac{1}{2} = c$$

$$a+2 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2} - 2 = -\frac{3}{2}$$

$$c = \frac{1}{2}$$

$$b \in \mathbb{R} \setminus \{0\}.$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\sqrt{1+bx}-1}{bx} \times \frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\cancel{1}+bx-\cancel{1}}{bx\left(\sqrt{1+bx}+1\right)}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\cancel{bx}}{\cancel{bx}\left(\sqrt{1+bx}+1\right)} \qquad b \neq 0$$

$$= \frac{1}{2}.$$

$f(1) = [1] = 1$

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore$ f is disc. at $x = 1$

$x = \frac{1}{2}$

$f(\tfrac{1}{2}) = [\tfrac{1}{2}] = 0$

$$\lim_{x \to \frac{1}{2}} f(x) = \lim_{x \to \frac{1}{2}} [x] = 0$$

f is cts at $x = \frac{1}{2}$.

Prob: Discuss the continuity of $f(x) = [x]$ at $x = 1, \frac{1}{2}$

$\rightarrow$ $x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{\substack{x \to 1 \\ x < 1}} [x] = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{\substack{x \to 1 \\ x > 1}} [x] = 1$$

→ let $x = n$ (an int.)

$$\lim_{x \to n-} f(x) = \lim_{\substack{x \to n \\ x < n}} [x] = n-1$$

$$\lim_{x \to n+} f(x) = \lim_{\substack{x \to n \\ x > n}} [x] = n$$

$$f(n) = [n] = n$$

$$\therefore \lim_{x \to n-} f(x) \neq \lim_{x \to n+} f(x)$$

$\therefore$ f is discts at $x = n$.

<u>Prob</u>: Discuss the continuity of $f(x) = [x]$.

→
$$f(x) = \begin{cases} \vdots & \vdots \\ -2 & -2 \le x < 1 \\ -1 & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$

$\therefore$ f is cts at $x = \alpha$.

Thus: f is discts at integral points and cts at non-integral points.

Let $x = \alpha \in \mathbb{R} \sim \mathbb{Z}$.

$\therefore \exists$ some int. n s.t

$$n < \alpha < n+1$$

$$\lim_{x \to \alpha} f(x) = \lim_{x \to \alpha} [x]$$
$$= n$$

$$f(\alpha) = [\alpha] = n$$

$$\therefore \lim_{x \to \alpha} f(x) = f(\alpha)$$

Let $x = n \in \mathbb{Z} \sim \{0\}$.

$$\lim_{x \to n-} f(x) = \lim_{\substack{x \to n \\ x < n}} x[x]$$

$$= \lim_{x \to n} (n-1) \cdot x$$

$$= n(n-1)$$

$$\lim_{x \to n+} f(x) = n^2$$

$$\lim_{x \to n+} f(x) \neq \lim_{x \to n-} f(x)$$

Prob: Discuss the continuity of $f(x) = x[x]$.

$\rightarrow$

$$f(x) = x[x] = \begin{cases} \vdots & \vdots \\ -2x, & -2 \leq x < -1 \\ -x, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ x, & 1 \leq x < 2 \\ 2x, & 2 \leq x < 3 \\ \vdots & \vdots \end{cases}$$

Let $x = \alpha \in \mathbb{R} \sim \mathbb{Z}$.

$\therefore \exists$ some int n s.t.

$$n < \alpha < n+1$$

$$\lim_{x \to \alpha} f(x) = \lim_{x \to \alpha} x[x]$$

$$= \lim_{x \to \alpha} nx$$

$$= n\alpha.$$

$$f(\alpha) = \alpha \cdot [\alpha] = n\alpha.$$

$\therefore f$ is cts at $x = \alpha$.

$\therefore f(x)$ is disc. at $x = n \in \mathbb{Z} \sim \{0\}$.

At $x = 0$

$$\lim_{x \to 0-} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} -x$$

$$= 0$$

$$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to 0 \\ x > 0}} 0 = 0$$

$$f(0) = 0$$

$\therefore f$ is cts at $x = 0$