KGS IAS

# Maths Optional

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$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By Ratio test,

$\sum u_n$ conv. for $\frac{1}{x} > 1$ i.e. $x < 1$

" div. " $\frac{1}{x} < 1$, i.e. $x > 1$

HW

Ans:

$x < 1$ conv.

$x > 1$ div.

$x = 1$, conv. for $\gamma > \alpha + \beta$

div. for $\gamma \leq \alpha + \beta$

Ignoring 1st term.

$$u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\cdot\beta\cdot(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots n\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)} x^n$$

$$= \frac{\left(1 + \frac{r+1}{n} + \frac{r}{n^2}\right)}{\left(1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right)}$$

$$= \left(1 + \frac{r+1}{n} + \frac{r}{n^2}\right)\left(1 + \underbrace{\frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}}\right)^{-1}$$

$$= \left(1 + \frac{r+1}{n} + \frac{r}{n^2}\right)\left[1 - \left(\frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right) + \left(\frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right)^2 - \right]$$

For $x = 1$, test fails.

For $x=1$, Applying Gauss's test

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{r}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$$

$$= \frac{1 + \frac{r}{n} + \frac{1}{n} + \frac{r}{n^2}}{1 + \frac{\beta}{n} + \frac{\alpha}{n} + \frac{\alpha\beta}{n^2}}$$

By Gauss's test.

$\sum u_n$ conv. iff $\gamma + \cancel{1} - (\alpha+\beta) > \cancel{1}$

i.e. $\gamma > \alpha + \beta$

$\sum u_n$ div. iff $\gamma + \cancel{1} - (\alpha+\beta) \leq \cancel{1}$

i.e. $\gamma \leqslant \alpha + \beta$.

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$$= 1 + \frac{1}{n}\left[-\alpha - \beta + \gamma + 1\right]$$

$$+ \frac{1}{n^2}\left[(\alpha+\beta)^2 - \alpha\beta - (\alpha+\beta)(\gamma+1) + \gamma\right]$$

+ ... terms cont. higher powers of $\frac{1}{n}$

$$= 1 + \frac{1}{n}\left[\gamma + 1 - (\alpha+\beta)\right] + \frac{1}{n^2}\left[\alpha^2 + \beta^2 + \alpha\beta - \alpha\gamma - \alpha - \beta\gamma - \beta + \gamma\right]$$

+ terms cont. higher powers of $\frac{1}{n}$.

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}- \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

<u>Leibnitz's test</u>: If the alternating series

$u_1 - u_2 + u_3 - u_4 - - -$

$(u_n > 0, \forall n)$ is s.t.

(i) $u_{n+1} \leq u_n$ $\forall n$.

(ii) $\lim u_n = 0$

## Series of abitrary terms

<u>Alternating series</u>: A series whose terms are alternately positive and -ve.

e.g

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} - - -$

is called an alternating series.

$\Rightarrow \frac{1}{(n+1)^p} < \frac{1}{n^p}$.

$\therefore u_{n+1} < u_n \ \forall n$.

$\lim u_n = \lim\limits_{n\to\infty} \frac{1}{n^p}$

$= 0, \quad \underline{p>0}$

$\therefore$ By Leibnitz's test

$\sum \frac{(-1)^{n-1}}{n^p}$ conv. for $p>0$.

then the series converge.

<u>Prob:</u> Show that the series

$\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} \cdots -$

$= \sum \frac{(-1)^{n-1}}{n^p}$ converges for $p>0$.

$\rightarrow$ Let $u_n = \frac{1}{n^p}$

$n+1 > n \quad \forall n$

$\Rightarrow (n+1)^p > n^p \quad \forall n \ \& \ p>0$

Ex: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\cdots$

$$= \sum \frac{(-1)^{n-1}}{n}.$$

→ By Leibnitz's test

This series is cgt.

But $1+\frac{1}{2}+\frac{1}{3}+\cdots$

$$= \sum \frac{1}{n} \text{ is div.}$$

∴ The series is cond. cgt.

Absolute convergence: A series $\sum u_n$ is said to be absolutely cgt iff $\sum |u_n|$ is cgt.

Conditional convergence:

A series which is convergent but not absolutely cgt is called conditionally cgt.

**Result:** Every absolutely convergent series is cgt.

**Proof:** Let $\sum u_n$ is an absolutely cgt series.

$\therefore \sum |u_n|$ is cgt.

For any $\varepsilon > 0$, by Cauchy's general principle of conv. $\exists$ a +ve int $m$ s.t.

**Ex:** $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} \cdots$ ——

$\longrightarrow \sum \frac{(-1)^{n-1}}{n^3}$ —— cgt

$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

$= \sum \frac{1}{n^3}$ —— cgt.

Absolutely cgt.

$\Rightarrow |u_{m+1} + \cdots + u_n| < \varepsilon \ \forall n \geq m$

$\therefore \sum u_n$ is cgt.

**Note:** $\sum \frac{(-1)^{n-1}}{n}$ is cgt.

Let $u_n = \frac{(-1)^{n-1}}{n}$.

$\sum u_n$ is cgt

But $\sum |u_n| = \sum \frac{1}{n}$ is dvgt.

$\big| |u_{m+1}| + |u_{m+2}| + \cdots + |u_n| \big| < \varepsilon$

$\forall n \geq m$

$\Rightarrow |u_{m+1}| + |u_{m+2}| + \cdots + |u_n| < \varepsilon$

$\forall n \geq m$.

Now

$|u_{m+1} + u_{m+2} + \cdots + u_n|$

$\leq |u_{m+1}| + |u_{m+2}| + \cdots + |u_n| < \varepsilon$

$\forall n \geq m$

$\rightarrow$ Let $u_n = \dfrac{\sin nx}{n^2}$

$$|u_n| = \left|\frac{\sin nx}{n^2}\right|$$

$$= \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}$$

By comparison test

$-1 \leq \sin\theta \leq 1$

i.e. $|\sin\theta| \leq 1$

$\sum |u_n|$ is cgt.

as $\sum \frac{1}{n^2}$ is cgt.

$\sum |u_n|$ is divergent, doesn't mean $\sum u_n$ is divergent.

Prob: Show that for any fixed value of $x$, the series $\sum \dfrac{\sin nx}{n^2}$ is cgt.

prob: Show that the series

$$x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

Converges absolutely for all values of $x$.

$\rightarrow$ Let $u_n = \dfrac{x^n}{\lfloor n}$.

$$\frac{|u_n|}{|u_{n+1}|} = \frac{\cancel{|x|^n}}{\cancel{\lfloor n}} \times \frac{\cancel{\lfloor n+1}^{\,n+1}}{\cancel{|x|^{n+1}}_{\,|x|}}$$

$\therefore \sum u_n$ is absolutely cgt.

$\therefore \sum u_n$ is cgt.

i.e. $\sum \dfrac{\sin nx}{n^2}$ is cgt.

$\therefore \sum |u_n|$ conv. $\forall x$.

$\Rightarrow \sum u_n$ conv. absolutely.

Note: $\boxed{\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0}$

$\therefore \lim \frac{|u_n|}{|u_{n+1}|} = \lim \frac{n+1}{|x|}$

$= \infty$ ($\forall x$ except $x = 0$)

$> 1$

$\therefore$ By Ratio test

$\sum |u_n|$ converges $\forall x$ except 0.

For $x = 0$, $\sum |u_n|$ obviously conv.

$$\lim \frac{|u_n|}{|u_{n+1}|} = \lim \left|\frac{n}{m-n}\right| \cdot \frac{1}{|x|}$$

$$= \lim \left|\frac{1}{\frac{m}{n}-1}\right| \cdot \frac{1}{|x|}$$

$$= \frac{1}{|x|}$$

By Ratio test $\sum |u_n|$ conv. if $\frac{1}{|x|} > 1$ i.e. $|x|<1$

$\therefore \sum u_n$ conv. abs.

**Prob:** Show that

$$\lim_{n\to\infty} \frac{m(m-1)\cdots(m-n+1)}{\lfloor n-1} \cdot x^n = 0$$

where $|x|<1$ and $m$ is any real number.

$\rightarrow$ Consider $\sum u_n$, where

$$u_n = \frac{m(m-1)\cdots(m-n+1)}{\lfloor n-1} x^n$$

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} \cdots -$$

$$\rightarrow \quad u_n = \frac{1}{\sqrt{2n-1}}$$

$$2n+1 > 2n-1$$

$$\Rightarrow \sqrt{2n+1} > \sqrt{2n-1}$$

$$\Rightarrow \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n-1}}$$

$$\Rightarrow u_{n+1} < u_n \ \forall n.$$

$\therefore \sum u_n$ conv.

$\therefore \lim u_n = 0$

i.e. $\lim \dfrac{m(m-1)\cdots(m-n+1)}{\lfloor n-1} x^n = 0$

when $|x| < 1$

__Proof:__ Test for convergence and absolute conv. of the series

$$1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+ - - -$$

$$\frac{1}{\sqrt{2n-1}} \sim \frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}\sqrt{n}}$$

Consider $v_n = \frac{1}{\sqrt{n}}$

$$\lim \frac{u_n}{v_n} = \lim \frac{\sqrt{n}}{\sqrt{2n-1}}$$

$$= \lim \frac{1}{\sqrt{2-\frac{1}{n}}} = \frac{1}{\sqrt{2}} \neq 0$$

$$\lim u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2n-1}}$$

$$= 0.$$

$\therefore$ By Leibnitz test

$\sum (-1)^{n-1} u_n$ conv

Now consider

$$\sum \left| (-1)^{n-1} u_n \right| = \sum u_n$$

Prob: Test for convergence of the series

$$\sum \frac{(-1)^n \cos n\alpha}{\sqrt{n^3}},$$

$\alpha$ being real.

$\rightarrow$ Let $u_n = \dfrac{(-1)^n \cdot \cos n\alpha}{\sqrt{n^3}}$

$$|u_n| = \left| \frac{(-1)^n \cdot \cos n\alpha}{n^{\frac{3}{2}}} \right| = \frac{|\cos n\alpha|}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}$$

$\therefore$ By Limit comp. test

$\sum u_n$ and $\sum v_n$ will have same nature.

$\sum v_n = \dfrac{1}{n^{\frac{1}{2}}}$ div.

$\therefore \sum u_n$ div.

$\therefore \sum (-1)^{n-1} u_n$ is not abs. cgt.

$\therefore$ is cond. cgt.

Prob: Discuss the convergence of the series

$$\sum \frac{\sin nx + \cos nx}{n^{3/2}}.$$

$\rightarrow$ Let $u_n = \dfrac{\sin nx + \cos nx}{n^{3/2}}$

$$|u_n| = \left| \frac{\sin nx + \cos nx}{n^{3/2}} \right|$$

$\therefore$ By comparison test.

$\sum |u_n|$ converges

as $\sum \dfrac{1}{n^{3/2}}$ conv.

$\therefore$ $\sum u_n$ conv. absolutely.

$\therefore$ $\sum u_n$ conv.

$\therefore$ By comparison test.

$\sum |u_n|$ converges as

$\sum \frac{1}{n^{3/2}}$ conv.

$\therefore$ $\sum u_n$ is abs. cgt.

$\therefore$ $\sum u_n$ is cgt.

$$= \frac{|\sin nx + \cos nx|}{n^{3/2}}$$

$$\leq \frac{|\sin nx|}{n^{3/2}} + \frac{|\cos nx|}{n^{3/2}}$$

$$\leq \frac{1}{n^{3/2}} + \frac{1}{n^{3/2}}$$

$\therefore$ $|u_n| \leq 2 \frac{1}{n^{\frac{3}{2}}}$ $\forall n$.

HW

<u>prob:</u> Test for absolute Convergence of the series

$$\sum (-1)^{n-1} \frac{n^2}{\lfloor n+1}$$