3-DIMENSTION

Vector Representation of a Point

Position vector of Point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$.

Distance Formula

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. $AB = |\overrightarrow{OB} - \overrightarrow{OA}|$

Distance of P from Coordinate Axes

PA = $\sqrt{y^2 + z^2}$, PB = $\sqrt{z^2 + x^2}$, PC = $\sqrt{x^2 + y^2}$

Section Formula

$x = \frac{mx_2+nx_1}{m+n}$, $y = \frac{my_2+ny_1}{m+n}$, $z = \frac{mz_2+nz_1}{m+n}$

Mid Point: $x = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$, $z = \frac{z_1+z_2}{2}$

Direction Cosines and Direction Ratios

(i) Direction cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

(iv) If ℓ, m, n the direction called the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2-x_1}{|PQ|}$, $m = \frac{y_2-y_1}{|PQ|}$ and $n = \frac{z_2-z_1}{|PQ|}$

Angle Between Two Line Segments

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

The line will be perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and

Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Projection of a Line Segment on a Line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the projection of PQ on a line having direction cosines ℓ, m, n is

$$|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

Equation of A Plane

General form: ax + by + cz + d = 0, where a, b, c are not all zero, a, b, c, d∈R

(i) Normal form: $\ell x + mx + nz = p$

(ii) Plane through the point (x_1, y_1, z_1):

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

(iii) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(iv) Vector form: $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(v) Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + λ = 0. Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(vi) Equation of a plane passing through a given point & parallel to the given vectors:

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

Or $\vec{r}.(\vec{b} \times \vec{c}) = \vec{a}.(\vec{b} \times \vec{c})$ (non parametric form)

A Plane & A Point

(i) Distance of the point (x′, y′, z′) from the plane ax + by + cz + d = 0 is given by $\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$

(ii) Length of the perpendicular from a point $(\vec{a})$ to plane $\vec{r}.\vec{n} = d$ is given by $p = \frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$.

(iii) Foot (x′, y′, z′) of perpendicular drawn from the point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 is given by $\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) To find image of a point w.r.t. a plane:

Let P(x_1, y_1, z_1) is a given point and ax + by + xz + d = 0is given plane Let (x′, y′, z′) is the image point. Then

$$\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

Angle Between Two Planes

(i) $\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$

(ii) Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(iii) The angle θ between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

(iv) Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$, λ is a scalar

Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1y + d_1 = 0$ and $a_2x + b_2y + c_2 + d_2 = 0$ are $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2x+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

(ii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \quad \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \quad \Rightarrow$ origin lies in acute angle

Family of Planes

(i) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(n_1 + \lambda\vec{n}_2) = d_1 + \lambda + d_2$ where λ is arbitrary scalar

Volume Of a Tetrahedron

Volume of a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Equation Of a Line

(i) A straight line is intersection of two planes.it is represented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + +c_2z + d_2 = 0$.

(ii) Symmetric form: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$.

(iii) Vector equation: $\vec{r} = \vec{a} + \lambda\vec{b}$

(iv) Reduction of cartesian form of equation of a line to vector form & vice versa

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$.

Angle Between a Plane and a Line

(i) If θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, then $\sin\,\theta = \left|\frac{a\ell+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right|$.

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\,\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b}\times\vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$, $\vec{b}\cdot\vec{n} = 0$

Condition For a Line to Lie in a Plane

(i) Cartesian form: Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ & $\vec{a}\cdot\vec{n} = d$

Skew Lines

(i) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. Lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$

If $\Delta = \begin{vmatrix} \alpha' - \alpha & \beta' - \beta & \gamma' - \gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(ii) Shortest distance formula for lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is $d = \left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

(iii) Vector Form: For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to be skew $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \neq 0$

(iv) Shortest distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2-\vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

(v) Condition of coplanarity of two lines $\vec{r} = \vec{a} + \lambda\vec{b}$ & $\vec{r} = \vec{c} + \mu\vec{d}$ is $[\vec{a} - \vec{c} \quad \vec{b} \quad \vec{d}] = 0$

Sphere

General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

$(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius of the sphere.