Appendix 1

PROOFS IN MATHEMATICS

❖ *Proofs are to Mathematics what calligraphy is to poetry. Mathematical works do consist of proofs just as poems do consist of characters.*
— VLADIMIR ARNOLD ❖

A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

Direct Proof It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

(i) Straight forward approach It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

Example 1 Show that if $x^2 - 5x + 6 = 0$, then $x = 3$ or $x = 2$.

Solution $x^2 - 5x + 6 = 0$ (given)

$\Rightarrow$ $(x - 3)(x - 2) = 0$ (replacing an expression by an equal/equivalent expression)

$\Rightarrow$ $x - 3 = 0$ or $x - 2 = 0$ (from the established theorem $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$, for a, b in **R**)

$\Rightarrow$ $x - 3 + 3 = 0 + 3$ or $x - 2 + 2 = 0 + 2$ (adding equal quantities on either side of the equation does not alter the nature of the equation)

$\Rightarrow$ $x + 0 = 3$ or $x + 0 = 2$ (using the identity property of integers under addition)

$\Rightarrow$ $x = 3$ or $x = 2$ (using the identity property of integers under addition)

Hence, $x^2 - 5x + 6 = 0$ implies $x = 3$ or $x = 2$.

Explanation Let p be the given statement "$x^2 - 5x + 6 = 0$" and q be the conclusion statement "$x = 3$ or $x = 2$".

From the statement p, we deduced the statement r : "$(x - 3)(x - 2) = 0$" by replacing the expression $x^2 - 5x + 6$ in the statement p by another expression $(x - 3)(x - 2)$ which is equal to $x^2 - 5x + 6$.

There arise two questions:

(i) How does the expression $(x - 3)(x - 2)$ is equal to the expression $x^2 - 5x + 6$?

(ii) How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x - 3) - 2(x - 3) = (x - 3)(x - 2)$.

The second one is by valid form of argumentation (rules of logic)

Next this statement r becomes premises or given and deduce the statement s " $x - 3 = 0$ or $x - 2 = 0$" and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that $p \Rightarrow q$ is true.

Starting with p, we deduce $p \Rightarrow r \Rightarrow s \Rightarrow \ldots \Rightarrow q$. This implies that "$p \Rightarrow q$" is true.

Example 2 Prove that the function f : $\mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution Note that a function f is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \text{(definition of one-one function)}$$

Now, given that $f(x_1) = f(x_2)$, i.e., $2x_1 + 5 = 2x_2 + 5$

$\Rightarrow$ $2x_1 + 5 - 5 = 2x_2 + 5 - 5$ (adding the same quantity on both sides)

$$\Rightarrow \quad 2x_1 + 0 = 2x_2 + 0$$

$$\Rightarrow \quad 2x_1 = 2x_2 \text{ (using additive identity of real number)}$$

$$\Rightarrow \quad \frac{2}{2}x_1 = \frac{2}{2}x_2 \text{ (dividing by the same non zero quantity)}$$

$$\Rightarrow \quad x_1 = x_2$$

Hence, the given function is one-one.

(ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset S of **N**, if

(i) the natural number $1 \in S$ and

(ii) the natural number $k + 1 \in S$ whenever $k \in S$, then S = **N**.

According to the principle of mathematical induction, if a statement "S(n) is true for $n = 1$" (or for some starting point j), and if "S(n) is true for $n = k$" implies that "S(n) is true for $n = k + 1$" (whatever integer $k \geq j$ may be), then the statement is true for any positive integer n, for all $n \geq j$.

We now consider some examples.

Example 3 Show that if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

Solution We have

$$P(n) : A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

We note that

$$P(1) : A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, P(1) is true.

Assume that P(k) is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

We want to prove that P(k + 1) is true whenever P(k) is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now $$A^{k+1} = A^k . A$$

Since P(k) is true, we have

$$A^{k+1} = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

(by matrix multiplication)

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, P(n) is true for all $n \geq 1$ (by the principle of mathematical induction).

(iii) Proof by cases or by exhaustion

This method of proving a statement $p \Rightarrow q$ is possible only when p can be split into several cases, r, s, t (say) so that $p = r \vee s \vee t$ (where "$\vee$" is the symbol for "OR").

If the conditionals $r \Rightarrow q$;

$s \Rightarrow q$;

and $t \Rightarrow q$

are proved, then $(r \vee s \vee t) \Rightarrow q$, is proved and so $p \Rightarrow q$ is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

Example 4 Show that in any triangle ABC,

$$a = b \cos C + c \cos B$$

Solution Let p be the statement "ABC is any triangle" and q be the statement "$a = b \cos C + c \cos B$"

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split p into three statements r, s and t, where

r : ABC is an acute angled triangle with $\angle$ C is acute.

s : ABC is an obtuse angled triangle with $\angle$ C is obtuse.

t : ABC is a right angled triangle with $\angle$ C is right angle.

Hence, we prove the theorem by three cases.

Case (i) When $\angle$ C is acute (Fig. A1.1).

From the right angled triangle ADB,

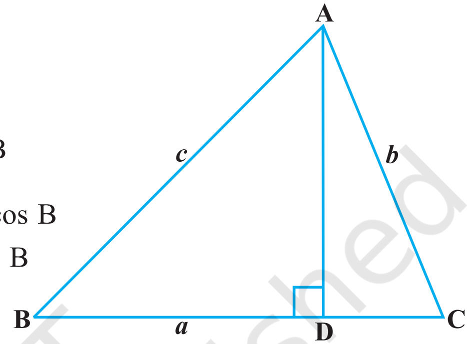

Fig A1.1

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B$$
$$= c \cos B$$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e. $$CD = AC \cos C$$
$$= b \cos C$$

Now $$a = BD + CD$$
$$= c \cos B + b \cos C \qquad \ldots (1)$$

Case (ii) When $\angle$ C is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B$$
$$= c \cos B$$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos \angle ACD$$
$$= \cos (180° – C)$$
$$= – \cos C$$

i.e. $$CD = – AC \cos C$$
$$= – b \cos C$$

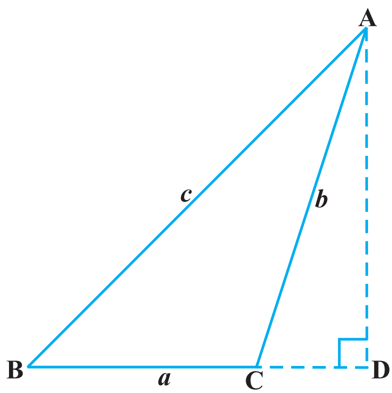

Fig A1.2

Now $a = \text{BC} = \text{BD} - \text{CD}$

i.e. $a = c \cos \text{B} - (-b \cos \text{C})$

$$a = c \cos \text{B} + b \cos \text{C} \quad \dots (2)$$

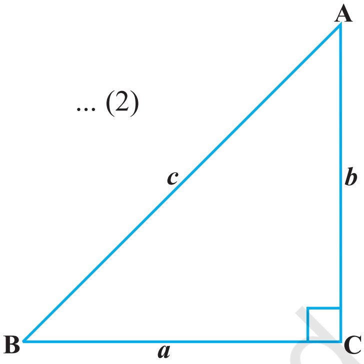

Fig A1.3

Case (iii) When ∠ C is a right angle (Fig A1.3).

From the right angled triangle ACB,

$$\frac{\text{BC}}{\text{AB}} = \cos \text{B}$$

i.e. $\text{BC} = \text{AB} \cos \text{B}$

$$a = c \cos \text{B},$$

and $b \cos \text{C} = b \cos 90^0 = 0.$

Thus, we may write $a = 0 + c \cos \text{B}$

$$= b \cos \text{C} + c \cos \text{B} \quad \dots (3)$$

From (1), (2) and (3). We assert that for any triangle ABC,

$$a = b \cos \text{C} + c \cos \text{B}$$

By case (i), $r \Rightarrow q$ is proved.

By case (ii), $s \Rightarrow q$ is proved.

By case (iii), $t \Rightarrow q$ is proved.

Hence, from the proof by cases, $(r \vee s \vee t) \Rightarrow q$ is proved, i.e., $p \Rightarrow q$ is proved.

Indirect Proof Instead of proving the given proposition directly, we establish the proof of the proposition through proving a proposition which is equivalent to the given proposition.

(i) Proof by contradiction (*Reductio Ad Absurdum*) : Here, we start with the assumption that the given statement is false. By rules of logic, we arrive at a conclusion contradicting the assumption and hence it is inferred that the assumption is wrong and hence the given statement is true.

Let us illustrate this method by an example.

Example 5 Show that the set of all prime numbers is infinite.

Solution Let P be the set of all prime numbers. We take the negation of the statement "the set of all prime numbers is infinite", i.e., we assume the set of all prime numbers to be finite. Hence, we can list all the prime numbers as $P_1, P_2, P_3, \dots, P_k$ (say). Note that we have assumed that there is no prime number other than $P_1, P_2, P_3, \dots, P_k$.

Now consider $N = (P_1 P_2 P_3 \dots P_k) + 1 \dots (1)$

N is not in the list as N is larger than any of the numbers in the list.

N is either prime or composite.

If N is a prime, then by (1), there exists a prime number which is not listed.

On the other hand, if N is composite, it should have a prime divisor. But none of the numbers in the list can divide N, because they all leave the remainder 1. Hence, the prime divisor should be other than the one in the list.

Thus, in both the cases whether N is a prime or a composite, we ended up with contradiction to the fact that we have listed all the prime numbers.

Hence, our assumption that set of all prime numbers is finite is false.

Thus, the set of all prime numbers is infinite.

Note Observe that the above proof also uses the method of proof by cases.

(ii) Proof by using contrapositive statement of the given statement

Instead of proving the conditional $p \Rightarrow q$, we prove its equivalent, i.e., $\sim q \Rightarrow \sim p$. (students can verify).

The contrapositive of a conditional can be formed by interchanging the conclusion and the hypothesis and negating both.

Example 6 Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Using this we have to show that "$2x_1 + 5 = 2x_2 + 5$" $\Rightarrow$ "$x_1 = x_2$". This is of the form $p \Rightarrow q$, where, p is $2x_1 + 5 = 2x_2 + 5$ and $q : x_1 = x_2$. We have proved this in Example 2 of "direct method".

We can also prove the same by using contrapositive of the statement. Now contrapositive of this statement is $\sim q \Rightarrow \sim p$, i.e., contrapositive of "if $f(x_1) = f(x_2)$, then $x_1 = x_2$" is "if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$".

Now $\qquad x_1 \neq x_2$

$\Rightarrow \qquad 2x_1 \neq 2x_2$

$\Rightarrow \qquad 2x_1 + 5 \neq 2x_2 + 5$

$\Rightarrow \qquad f(x_1) \neq f(x_2)$.

Since "$\sim q \Rightarrow \sim p$", is equivalent to "$p \Rightarrow q$" the proof is complete.

Example 7 Show that "if a matrix A is invertible, then A is non singular".

Solution Writing the above statement in symbolic form, we have
$p \Rightarrow q$, where, p is "matrix A is invertible" and q is "A is non singular"

Instead of proving the given statement, we prove its contrapositive statement, i.e., if A is not a non singular matrix, then the matrix A is not invertible.

If A is not a non singular matrix, then it means the matrix A is singular, i.e.,

$$|A| = 0$$

Then $$A^{-1} = \frac{adj\ A}{|A|} \text{ does not exist as } |A| = 0$$

Hence, A is not invertible.

Thus, we have proved that if A is not a non singular matrix, then A is not invertible. i.e., $\sim q \Rightarrow \sim p$.

Hence, if a matrix A is invertible, then A is non singular.

(iii) Proof by a counter example

In the history of Mathematics, there are occasions when all attempts to find a valid proof of a statement fail and the uncertainty of the truth value of the statement remains unresolved.

In such a situation, it is beneficial, if we find an example to falsify the statement. The example to disprove the statement is called a *counter example*. Since the disproof of a proposition $p \Rightarrow q$ is merely a proof of the proposition $\sim (p \Rightarrow q)$. Hence, this is also a method of proof.

Example 8 For each $n, 2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$.

This was once thought to be true on the basis that

$$2^{2^1} + 1 = 2^2 + 1 = 5 \text{ is a prime.}$$

$$2^{2^2} + 1 = 2^4 + 1 = 17 \text{ is a prime.}$$

$$2^{2^3} + 1 = 2^8 + 1 = 257 \text{ is a prime.}$$

However, at first sight the generalisation looks to be correct. But, eventually it was shown that $$2^{2^5} + 1 = 2^{32} + 1 = 4294967297$$

which is not a prime since 4294967297 = 641 × 6700417 (a product of two numbers).

So the generalisation "For each n, $2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$" is false.

Just this one example $2^{2^5} + 1$ is sufficient to disprove the generalisation. This is the counter example.

Thus, we have proved that the generalisation "For each $n, 2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$" is not true in general.

Example 9 Every continuous function is differentiable.

Proof We consider some functions given by

(i) $f(x) = x^2$

(ii) $g(x) = e^x$

(iii) $h(x) = \sin x$

These functions are continuous for all values of x. If we check for their differentiability, we find that they are all differentiable for all the values of x. This makes us to believe that the generalisation "Every continuous function is differentiable" may be true. But if we check the differentiability of the function given by "$\phi(x) = |x|$" which is continuous, we find that it is not differentiable at $x = 0$. This means that the statement "Every continuous function is differentiable" is false, in general. Just this one function "$\phi(x) = |x|$" is sufficient to disprove the statement. Hence, "$\phi(x) = |x|$" is called a counter example to disprove "Every continuous function is differentiable".

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling.* Let us consider the following problems:

(i) To find the width of a river (particularly, when it is difficult to cross the river).

(ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).

(iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).

(iv) To find the temperature at the surface of the Sun.

(v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).

(vi) To find the mass of the Earth.

(vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).

(viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).

(ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and infact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

A.2.3 Principles of Mathematical Modelling

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

(i) Identify the need for the model. (for what we are looking for)

(ii) List the parameters/variables which are required for the model.

(iii) Identify the available relevent data. (what is given?)

(iv) Identify the circumstances that can be applied (assumptions)

(v) Identify the governing physical principles.

(vi) Identify
   (a) the equations that will be used.
   (b) the calculations that will be made.
   (c) the solution which will follow.

(vii) Identify tests that can check the
   (a) consistency of the model.
   (b) utility of the model.

(viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

Step 1: Identify the physical situation.

Step 2: Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

Step 3: Find the solution of the mathematical problem.

Step 4: Interpret the result in terms of the original problem and compare the result with observations or experiments.

Step 5: If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

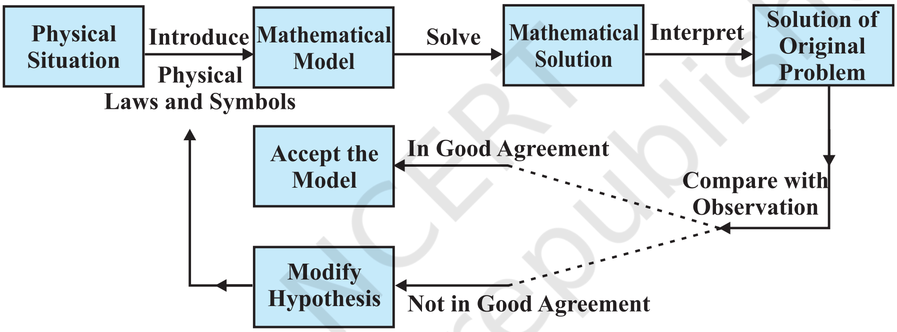

Fig A.2.1

Example 1 Find the height of a given tower using mathematical modelling.

Solution Step 1 Given physical situation is "to find the height of a given tower".

Step 2 Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let PQ = h and let height of tower be H. Let α be the angle of elevation from the eye of the observer to the top of the tower.

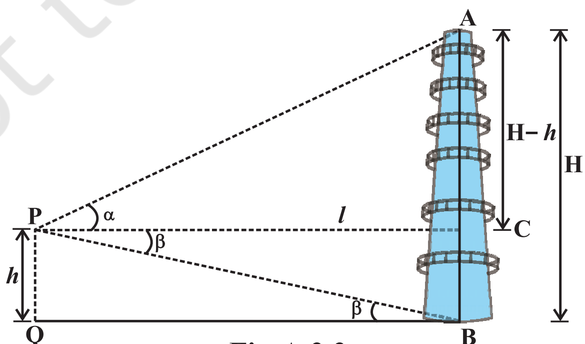

Fig A.2.2

Let $l = \text{PC} = \text{QB}$

Now $$\tan\alpha = \frac{\text{AC}}{\text{PC}} = \frac{\text{H}-h}{l}$$

or $$\text{H} = h + l\tan\alpha \qquad \dots (1)$$

Step 3 Note that the values of the parameters h, l and α (using sextant) are known to the observer and so (1) gives the solution of the problem.

Step 4 In case, if the foot of the tower is not accessible, i.e., when l is not known to the observer, let β be the angle of depression from P to the foot B of the tower. So from ΔPQB, we have

$$\tan\beta = \frac{\text{PQ}}{\text{QB}} = \frac{h}{l} \text{ or } l = h\cot\beta$$

Step 5 is not required in this situation as exact values of the parameters h, l, α and β are known.

Example 2 Let a business firm produces three types of products P_1, P_2 and P_3 that uses three types of raw materials R_1, R_2 and R_3. Let the firm has purchase orders from two clients F_1 and F_2. Considering the situation that the firm has a limited quantity of R_1, R_2 and R_3, respectively, prepare a model to determine the quantities of the raw material R_1, R_2 and R_3 required to meet the purchase orders.

Solution Step 1 The physical situation is well identified in the problem.

Step 2 Let A be a matrix that represents purchase orders from the two clients F_1 and F_2. Then, A is of the form

$$A = \begin{array}{c} \\ F_1 \\ F_2 \end{array}\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Let B be the matrix that represents the amount of raw materials R_1, R_2 and R_3, required to manufacture each unit of the products P_1, P_2 and P_3. Then, B is of the form

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Step 3 Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$AB = \begin{matrix} & R_1 & R_2 & R_3 \\ F_1 & \bullet & \bullet & \bullet \\ F_2 & \bullet & \bullet & \bullet \end{matrix}$$

which in fact gives the desired quantities of the raw materials R_1, R_2 and R_3 to fulfill the purchase orders of the two clients F_1 and F_2.

Example 3 Interpret the model in Example 2, in case

$$A = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix},\ B = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of R_1, 455 units of R_2 and 140 units of R_3.

Solution Note that

$$AB = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

$$= \begin{matrix} & R_1 & R_2 & R_3 \\ F_1 & 165 & 247 & 87 \\ F_2 & 170 & 220 & 60 \end{matrix}$$

This clearly shows that to meet the purchase order of F_1 and F_2, the raw material required is 335 units of R_1, 467 units of R_2 and 147 units of R_3 which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

Remark If we replace A in Example 3 by A_1 given by

$$A_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$A_1 B = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} = \begin{bmatrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{bmatrix}$$

This requires 311 units of R_1, 436 units of R_2 and 138 units of R_3 which are well below the available raw materials, i.e., 330 units of R_1, 455 units of R_2 and 140 units of R_3. Thus, if the revised purchase orders of the clients are given by A_1, then the firm can easily supply the purchase orders of the two clients.

Note One may further modify A so as to make full use of the available raw material.

Query Can we make a mathematical model with a given B and with fixed quantities of the available raw material that can help the firm owner to ask the clients to modify their orders in such a way that the firm makes the full use of its available raw material?

The answer to this query is given in the following example:

Example 4 Suppose P_1, P_2, P_3 and R_1, R_2, R_3 are as in Example 2. Let the firm has 330 units of R_1, 455 units of R_2 and 140 units of R_3 available with it and let the amount of raw materials R_1, R_2 and R_3 required to manufacture each unit of the three products is given by

$$B = \begin{matrix} & \begin{matrix} R_1 & R_2 & R_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \end{matrix}$$

How many units of each product is to be made so as to utilise the full available raw material?

Solution Step 1 The situation is easily identifiable.

Step 2 Suppose the firm produces x units of P_1, y units of P_2 and z units of P_3. Since product P_1 requires 3 units of R_1, P_2 requires 7 units of R_1 and P_3 requires 5 units of R_1 (observe matrix B) and the total number of units, of R_1, available is 330, we have

$$3x + 7y + 5z = 330 \text{ (for raw material } R_1)$$

Similarly, we have

$$4x + 9y + 12z = 455 \text{ (for raw material } R_2)$$

and

$$3y + 7z = 140 \text{ (for raw material } R_3)$$

This system of equations can be expressed in matrix form as

$$\begin{bmatrix} 3 & 7 & 5 \\ 4 & 9 & 12 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 330 \\ 455 \\ 140 \end{bmatrix}$$

Step 3 Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 5 \end{bmatrix}$$

This gives $x = 20$, $y = 35$ and $z = 5$. Thus, the firm can produce 20 units of P_1, 35 units of P_2 and 5 units of P_3 to make full use of its available raw material.

Remark One may observe that if the manufacturer decides to manufacture according to the available raw material and not according to the purchase orders of the two clients F_1 and F_2 (as in Example 3), he/she is unable to meet these purchase orders as F_1 demanded 6 units of P_3 where as the manufacturer can make only 5 units of P_3.

Example 5 A manufacturer of medicines is preparing a production plan of medicines M_1 and M_2. There are sufficient raw materials available to make 20000 bottles of M_1 and 40000 bottles of M_2, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of M_1, it takes 1 hour to prepare enough material to fill 1000 bottles of M_2 and there are 66 hours available for this operation. The profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2. How should the manufacturer schedule his/her production in order to maximise profit?

Solution Step 1 To find the number of bottles of M_1 and M_2 in order to maximise the profit under the given hypotheses.

Step 2 Let x be the number of bottles of type M_1 medicine and y be the number of bottles of type M_2 medicine. Since profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2, therefore the objective function (which is to be maximised) is given by

$$Z \equiv Z(x, y) = 8x + 7y$$

The objective function is to be maximised subject to the constraints (Refer Chapter 12 on Linear Programming)

$$\left.\begin{aligned} x &\leq 20000 \\ y &\leq 40000 \\ x + y &\leq 45000 \\ 3x + y &\leq 66000 \\ x \geq 0, y &\geq 0 \end{aligned}\right\} \quad \dots (1)$$

Step 3 The shaded region OPQRST is the feasible region for the constraints (1) (Fig A.2.3). The co-ordinates of vertices O, P, Q, R, S and T are (0, 0), (20000, 0), (20000, 6000), (10500, 34500), (5000, 40000) and (0, 40000), respectively.

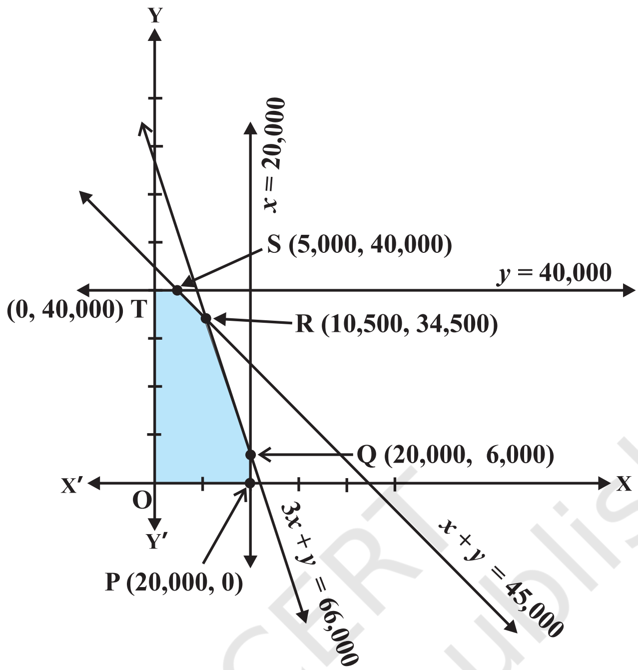

Fig A.2.3

Note that

$$\text{Z at P } (0, 0) = 0$$
$$\text{Z at P } (20000, 0) = 8 \times 20000 = 160000$$
$$\text{Z at Q } (20000, 6000) = 8 \times 20000 + 7 \times 6000 = 202000$$
$$\text{Z at R } (10500, 34500) = 8 \times 10500 + 7 \times 34500 = 325500$$
$$\text{Z at S} = (5000, 40000) = 8 \times 5000 + 7 \times 40000 = 320000$$
$$\text{Z at T} = (0, 40000) = 7 \times 40000 = 280000$$

Now observe that the profit is maximum at x = 10500 and y = 34500 and the maximum profit is ₹325500. Hence, the manufacturer should produce 10500 bottles of M_1 medicine and 34500 bottles of M_2 medicine in order to get maximum profit of ₹325500.

Example 6 Suppose a company plans to produce a new product that incur some costs (fixed and variable) and let the company plans to sell the product at a fixed price. Prepare a mathematical model to examine the profitability.

Solution Step 1 Situation is clearly identifiable.

Step 2 Formulation: We are given that the costs are of two types: fixed and variable. The fixed costs are independent of the number of units produced (e.g., rent and rates), while the variable costs increase with the number of units produced (e.g., material). Initially, we assume that the variable costs are directly proportional to the number of units produced — this should simplify our model. The company earn a certain amount of money by selling its products and wants to ensure that it is maximum. For convenience, we assume that all units produced are sold immediately.

The mathematical model

Let x = number of units produced and sold

C = total cost of production (in rupees)

I = income from sales (in rupees)

P = profit (in rupees)

Our assumptions above state that C consists of two parts:

(i) fixed cost = a (in rupees),

(ii) variable cost = b (rupees/unit produced).

Then $$C = a + bx \qquad \text{... (1)}$$

Also, income I depends on selling price s (rupees/unit)

Thus $$I = sx \qquad \text{... (2)}$$

The profit P is then the difference between income and costs. So

$$\begin{aligned} P &= I - C \\ &= sx - (a + bx) \\ &= (s - b)\, x - a \end{aligned} \qquad \text{... (3)}$$

We now have a mathematical model of the relationships (1) to (3) *between the variables* x, C, I, P, a, b, s. These variables may be classified as:

independent	x
dependent	C, I, P
parameters	a, b, s

The manufacturer, knowing x, a, b, s can determine P.

Step 3 From (3), we can observe that for the break even point (i.e., make neither profit nor loss), he must have P = 0, i.e., $x = \dfrac{a}{s-b}$ units.

Steps 4 and 5 In view of the break even point, one may conclude that if the company produces few units, i.e., less than $x = \dfrac{a}{s-b}$ units, then the company will suffer loss and if it produces large number of units, i.e., much more than $\frac{a}{s-b}$ units, then it can make huge profit. Further, if the break even point proves to be unrealistic, then another model could be tried or the assumptions regarding cash flow may be modified.

Remark From (3), we also have

$$\frac{dP}{dx} = s - b$$

This means that rate of change of P with respect to x depends on the quantity $s - b$, which is the difference of selling price and the variable cost of each product. Thus, in order to gain profit, this should be positive and to get large gains, we need to produce large quantity of the product and at the same time try to reduce the variable cost.

Example 7 Let a tank contains 1000 litres of brine which contains 250 g of salt per litre. Brine containing 200 g of salt per litre flows into the tank at the rate of 25 litres per minute and the mixture flows out at the same rate. Assume that the mixture is kept uniform all the time by stirring. What would be the amount of salt in the tank at any time t?

Solution Step 1 The situation is easily identifiable.

Step 2 Let $y = y(t)$ denote the amount of salt (in kg) in the tank at time t (in minutes) after the inflow, outflow starts. Further assume that y is a differentiable function.

When $t = 0$, i.e., before the inflow–outflow of the brine starts,

$$y = 250 \text{ g} \times 1000 = 250 \text{ kg}$$

Note that the change in y occurs due to the inflow, outflow of the mixture.

Now the inflow of brine brings salt into the tank at the rate of 5 kg per minute (as 25 × 200 g = 5 kg) and the outflow of brine takes salt out of the tank at the rate of

$25\left(\frac{y}{1000}\right) = \frac{y}{40}$ kg per minute (as at time t, the salt in the tank is $\frac{y}{1000}$ kg).

Thus, the rate of change of salt with respect to t is given by

$$\frac{dy}{dt} = 5 - \frac{y}{40} \qquad \text{(Why?)}$$

or

$$\frac{dy}{dt} + \frac{1}{40}y = 5 \qquad \text{... (1)}$$

This gives a mathematical model for the given problem.

Step 3 Equation (1) is a linear equation and can be easily solved. The solution of (1) is given by

$$y e^{\frac{t}{40}} = 200 e^{\frac{t}{40}} + \text{C} \text{ or } y(t) = 200 + \text{C } e^{-\frac{t}{40}} \quad \text{... (2)}$$

where, c is the constant of integration.

Note that when $t = 0$, $y = 250$. Therefore, $250 = 200 + \text{C}$

or $\text{C} = 50$

Then (2) reduces to

$$y = 200 + 50\, e^{-\frac{t}{40}} \quad \text{... (3)}$$

or $$\frac{y-200}{50} = e^{-\frac{t}{40}}$$

or $$e^{\frac{t}{40}} = \frac{50}{y-200}$$

Therefore $$t = 40 \log_e \left(\frac{50}{y-200}\right) \quad \text{... (4)}$$

Here, the equation (4) gives the time t at which the salt in tank is y kg.

Step 4 Since $e^{-\frac{t}{40}}$ is always positive, from (3), we conclude that $y > 200$ at all times Thus, the minimum amount of salt content in the tank is 200 kg.

Also, from (4), we conclude that $t > 0$ if and only if $0 < y - 200 < 50$ i.e., if and only if $200 < y < 250$ i.e., the amount of salt content in the tank after the start of inflow and outflow of the brine is between 200 kg and 250 kg.

Limitations of Mathematical Modelling

Till today many mathematical models have been developed and applied successfully to understand and get an insight into thousands of situations. Some of the subjects like mathematical physics, mathematical economics, operations research, bio-mathematics etc. are almost synonymous with mathematical modelling.

But there are still a large number of situations which are yet to be modelled. The reason behind this is that either the situation are found to be very complex or the mathematical models formed are mathematically intractable.

The development of the powerful computers and super computers has enabled us to mathematically model a large number of situations (even complex situations). Due to these fast and advanced computers, it has been possible to prepare more realistic models which can obtain better agreements with observations.

However, we do not have good guidelines for choosing various parameters / variables and also for estimating the values of these parameters / variables used in a mathematical model. Infact, we can prepare reasonably accurate models to fit any data by choosing five or six parameters / variables. We require a minimal number of parameters / variables to be able to estimate them accurately.

Mathematical modelling of large or complex situations has its own special problems. These type of situations usually occur in the study of world models of environment, oceanography, pollution control etc. Mathematical modellers from all disciplines — mathematics, computer science, physics, engineering, social sciences, etc., are involved in meeting these challenges with courage.

ANSWERS

EXERCISE 1.1

1. (i) Neither reflexive nor symmetric nor transitive.
   (ii) Neither reflexive nor symmetric but transitive.
   (iii) Reflexive and transitive but not symmetric.
   (iv) Reflexive, symmetric and transitive.
   (v) (a) Reflexive, symmetric and transitive.
       (b) Reflexive, symmetric and transitive.
       (c) Neither reflexive nor symmetric nor transitive.
       (d) Neither reflexive nor symmetric but transitive.
       (e) Neither reflexive nor symmetric nor transitive.
3. Neither reflexive nor symmetric nor transitive.
5. Neither reflexive nor symmetric nor transitive.
9. (i) {1, 5, 9}, (ii) {1}
12. T_1 is related to T_3.
13. The set of all triangles
14. The set of all lines $y = 2x + c$, $c \in \mathbf{R}$
15. B
16. C

EXERCISE 1.2

1. No
2. (i) Injective but not surjective
   (ii) Neither injective nor surjective
   (iii) Neither injective nor surjective
   (iv) Injective but not surjective
   (v) Injective but not surjective
7. (i) One-one and onto
   (ii) Neither one-one nor onto.
9. No
10. Yes
11. D
12. A

Miscellaneous Exercise on Chapter 1

3. No
4. $n!$
5. Yes
6. A
7. B

EXERCISE 2.1

1. $\frac{-\pi}{6}$ 2. $\frac{\pi}{6}$ 3. $\frac{\pi}{6}$ 4. $\frac{-\pi}{3}$

5. $\frac{2\pi}{3}$ 6. $-\frac{\pi}{4}$ 7. $\frac{\pi}{6}$ 8. $\frac{\pi}{6}$

9. $\frac{3\pi}{4}$ 10. $\frac{-\pi}{4}$ 11. $\frac{3\pi}{4}$ 12. $\frac{2\pi}{3}$

13. B 14. B

EXERCISE 2.2

3. $\frac{1}{2}\tan^{-1} x$ 4. $\frac{x}{2}$ 5. $\frac{\pi}{4} - x$

6. $\sin^{-1}\frac{x}{a}$ 7. $3\tan^{-1}\frac{x}{a}$ 8. $\frac{\pi}{4}$

9. $\frac{x+y}{1-xy}$ 10. $\frac{\pi}{3}$

11. $\frac{-\pi}{4}$ 12. $\frac{17}{6}$ 13. B 14. D

15. B

Miscellaneous Exercise on Chapter 2

1. $\frac{\pi}{6}$ 2. $\frac{\pi}{6}$ 11. $x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$ 12. $x = \frac{1}{\sqrt{3}}$

13. D 14. C

EXERCISE 3.1

1. (i) 3×4 (ii) 12 (iii) $19, 35, -5, 12, \frac{5}{2}$
2. $1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, 6 \times 4, 8 \times 3, 12 \times 2, 24 \times 1; 1 \times 13, 13 \times 1$
3. $1 \times 18, 2 \times 9, 3 \times 6, 6 \times 3, 9 \times 2, 18 \times 1; 1 \times 5, 5 \times 1$
4. (i) $\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

5. (i) $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

6. (i) $x = 1, \quad y = 4, \quad z = 3$

(ii) $x = 4, \quad y = 2, \quad z = 0$ or $x = 2, \quad y = 4, \; z = 0$

(iii) $x = 2, \quad y = 4, \quad z = 3$

7. $a = 1, b = 2, c = 3, d = 4$

8. C **9.** B **10.** D

EXERCISE 3.2

1. (i) $\text{A} + \text{B} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $\text{A} - \text{B} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(iii) $3\text{A} - \text{C} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$ (iv) $\text{AB} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$ (v) $\text{BA} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

2. (i) $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$ (ii) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(iii) $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. (i) $\begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

4. $\text{A} + \text{B} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$, $\text{B} - \text{C} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. (i) $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (ii) $X = \begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix}$, $Y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

8. $X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ **9.** $x = 3, y = 3$ **10.** $x = 3, y = 6, z = 9, t = 6$

11. $x = 3, y = -4$ **12.** $x = 2, y = 4, w = 3, z = 1$

15. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ **17.** $k = 1$

19. (a) ₹15000, ₹15000 (b) ₹5000, ₹25000

20. ₹20160 **21.** A **22.** B

EXERCISE 3.3

1. (i) $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. (i) $A = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & -2 & -2 \\ \frac{-5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ \frac{-5}{2} & 0 & 3 \\ \frac{-3}{2} & -3 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

11. A **12.** B

EXERCISE 3.4

1. D

Miscellaneous Exercise on Chapter 3

3. $x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$

4. $x = -1$ **6.** $x = \pm 4\sqrt{3}$

7. (a) Total revenue in the market - I = ₹ 46000

Total revenue in the market - II = ₹ 53000

(b) ₹ 15000, ₹ 17000

8. $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$ **9.** C **10.** B **11.** C

EXERCISE 4.1

1. (i) 18 **2.** (i) 1, (ii) $x^3 - x^2 + 2$

5. (i) – 12, (ii) 46, (iii) 0, (iv) 5 **6.** 0

7. (i) $x = \pm\sqrt{3}$, (ii) $x = 2$ **8.** (B)

EXERCISE 4.2

1. (i) $\frac{15}{2}$, (ii) $\frac{47}{2}$, (iii) 15

3. (i) 0, 8, (ii) 0, 8 **4.** (i) $y = 2x$, (ii) $x - 3y = 0$ **5.** (D)

EXERCISE 4.3

1. (i) $M_{11} = 3, M_{12} = 0, M_{21} = -4, M_{22} = 2, A_{11} = 3, A_{12} = 0, A_{21} = 4, A_{22} = 2$

(ii) $M_{11} = d, M_{12} = b, M_{21} = c, M_{22} = a$

$A_{11} = d, A_{12} = -b, A_{21} = -c, A_{22} = a$

2. (i) $M_{11} = 1, M_{12} = 0, M_{13} = 0, M_{21} = 0, M_{22} = 1, M_{23} = 0, M_{31} = 0, M_{32} = 0, M_{33} = 1,$

$A_{11} = 1, A_{12} = 0, A_{13} = 0, A_{21} = 0, A_{22} = 1, A_{23} = 0, A_{31} = 0, A_{32} = 0, A_{33} = 1$

(ii) $M_{11} = 11, M_{12} = 6, M_{13} = 3, M_{21} = -4, M_{22} = 2, M_{23} = 1, M_{31} = -20, M_{32} = -13, M_{33} = 5$

$A_{11} = 11, A_{12} = -6, A_{13} = 3, A_{21} = 4, A_{22} = 2, A_{23} = -1, A_{31} = -20, A_{32} = 13, A_{33} = 5$

3. 7 **4.** $(x - y)(y - z)(z - x)$ **5.** (D)

EXERCISE 4.4

1. $\begin{matrix} 4 & -2 \\ -3 & 1 \end{matrix}$ **2.** $\begin{matrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{matrix}$ **5.** $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6. $\frac{1}{13}\begin{matrix} 2 & -5 \\ 3 & -1 \end{matrix}$ **7.** $\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ **8.** $\frac{-1}{3}\begin{matrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{matrix}$

9. $\frac{-1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$ **10.** $\begin{matrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{matrix}$ **11.** $\begin{matrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{matrix}$

13. $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ **14.** $a = -4, b = 1$ **15.** $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

16. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **17.** B **18.** B

EXERCISE 4.5

1. Consistent
2. Consistent
3. Inconsistent
4. Consistent
5. Inconsistent
6. Consistent
7. $x = 2, y = -3$
8. $x = \frac{-5}{11}, y = \frac{12}{11}$
9. $x = \frac{-6}{11}, y = \frac{-19}{11}$
10. $x = -1, y = 4$
11. $x = 1, y = \frac{1}{2}, z = \frac{-3}{2}$
12. $x = 2, y = -1, z = 1$
13. $x = 1, y = 2, z = -1$
14. $x = 2, y = 1, z = 3$
15. $\begin{matrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{matrix}$, $x = 1, y = 2, z = 3$
16. cost of onions per kg = ₹ 5
    cost of wheat per kg = ₹ 8
    cost of rice per kg = ₹ 8

Miscellaneous Exercise on Chapter 4

2. 1
3. $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
5. $-2(x^3 + y^3)$
6. xy
7. $x = 2, y = 3, z = 5$
8. A
9. D

EXERCISE 5.1

2. f is continuous at $x = 3$
3. (a), (b), (c) and (d) are all continuous functions
5. f is continuous at $x = 0$ and $x = 2$; Not continuous at $x = 1$
6. Discontinuous at $x = 2$
7. Discontinuous at $x = 3$

8. Discontinuous at $x = 0$

9. No point of discontinuity

10. No point of discontinuity

11. No point of discontinuity

12. f is discontinuous at $x = 1$

13. f is not continuous at $x = 1$

14. f is not continuous at $x = 1$ and $x = 3$

15. $x = 1$ is the only point of discontinuity

16. Continuous

17. $a = b + \frac{2}{3}$

18. For no value of λ, f is continuous at $x = 0$ but f is continuous at $x = 1$ for any value of λ.

20. f is continuous at $x = \pi$

21. (a), (b) and (c) are all continuous

22. Cosine function is continuous for all $x \in \mathbf{R}$; cosecant is continuous except for $x = n\pi$, $n \in \mathbf{Z}$; secant is continuous except for $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbf{Z}$ and cotangent function is continuous except for $x = n\pi$, $n \in \mathbf{Z}$

23. There is no point of discontinuity.

24. Yes, f is continuous for all $x \in \mathbf{R}$

25. f is continuous for all $x \in \mathbf{R}$

26. $k = 6$

27. $k = \frac{3}{4}$

28. $k = \frac{-2}{\pi}$

29. $k = \frac{9}{5}$

30. $a = 2$, $b = 1$

34. There is no point of discontinuity.

EXERCISE 5.2

1. $2x \cos(x^2 + 5)$

2. $-\cos x \sin(\sin x)$

3. $a \cos(ax + b)$

4. $\dfrac{\sec(\tan\sqrt{x}).\tan(\tan\sqrt{x}).\sec^2\sqrt{x}}{2\sqrt{x}}$

5. $a \cos(ax + b) \sec(cx + d) + c \sin(ax + b) \tan(cx + d) \sec(cx + d)$

6. $10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2 x^5$

7. $\dfrac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$

8. $-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

EXERCISE 5.3

1. $\dfrac{\cos x-2}{3}$
2. $\dfrac{2}{\cos y-3}$
3. $-\dfrac{a}{2by+\sin y}$
4. $\dfrac{\sec^2 x-y}{x+2y-1}$
5. $-\dfrac{(2x+y)}{(x+2y)}$
6. $-\dfrac{(3x^2+2xy+y^2)}{(x^2+2xy+3y^2)}$
7. $\dfrac{y\sin xy}{\sin 2y-x\sin xy}$
8. $\dfrac{\sin 2x}{\sin 2y}$
9. $\dfrac{2}{1+x^2}$
10. $\dfrac{3}{1+x^2}$
11. $\dfrac{2}{1+x^2}$
12. $\dfrac{-2}{1+x^2}$
13. $\dfrac{-2}{1+x^2}$
14. $\dfrac{2}{\sqrt{1-x^2}}$
15. $-\dfrac{2}{\sqrt{1-x^2}}$

EXERCISE 5.4

1. $\dfrac{e^x(\sin x-\cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$
2. $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x\in(-1,1)$
3. $3x^2e^{x^3}$
4. $-\dfrac{e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}$
5. $-e^x\tan e^x,\ e^x \neq (2n+1)\dfrac{\pi}{2}, n\in\mathbf{N}$
6. $e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$
7. $\dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x>0$
8. $\dfrac{1}{x\log x}, x>1$
9. $-\dfrac{(x\sin x\cdot\log x+\cos x)}{x(\log x)^2}, x>0$
10. $-\left(\dfrac{1}{x}+e^x\right)\sin(\log x+e^x), x>0$

EXERCISE 5.5

1. – cos x cos $2x$ cos $3x$ [tan x + 2 tan $2x$ + 3 tan $3x$]
2. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}-\dfrac{1}{x-5}\right]$

3. $(\log x)^{\cos x}\left[\dfrac{\cos x}{x\log x}-\sin x\log(\log x)\right]$

4. $x^x(1+\log x)-2^{\sin x}\cos x\log 2$

5. $(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

6. $\left(x+\dfrac{1}{x}\right)^x\left[\dfrac{x^2-1}{x^2+1}+\log\left(x+\dfrac{1}{x}\right)\right]+x^{1+\frac{1}{x}}\left(\dfrac{x+1-\log x}{x^2}\right)$

7. $(\log x)^{x-1}[1+\log x\,.\,\log(\log x)]+2x^{\log x-1}\,.\,\log x$

8. $(\sin x)^x(x\cot x+\log\sin x)+\dfrac{1}{2}\dfrac{1}{\sqrt{x-x^2}}$

9. $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]+(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]$

10. $x^{x\cos x}[\cos x\,.\,(1+\log x)-x\sin x\log x]-\dfrac{4x}{(x^2-1)^2}$

11. $(x\cos x)^x[1-x\tan x+\log(x\cos x)]+(x\sin x)^{\frac{1}{x}}\left[\dfrac{x\cot x+1-\log(x\sin x)}{x^2}\right]$

12. $-\dfrac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

13. $\dfrac{y}{x}\left(\dfrac{y-x\log y}{x-y\log x}\right)$

14. $\dfrac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$

15. $\dfrac{y(x-1)}{x(y+1)}$

16. $(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}+\dfrac{4x^3}{1+x^4}+\dfrac{8x^7}{1+x^8}\right]$; $f'(1)=120$

17. $5x^4-20x^3+45x^2-52x+11$

EXERCISE 5.6

1. t^2

2. $\dfrac{b}{a}$

3. $-4\sin t$

4. $-\dfrac{1}{t^2}$

5. $\dfrac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$

6. $-\cot\dfrac{\theta}{2}$

7. $-\cot 3t$

8. $\tan t$

9. $\dfrac{b}{a}\operatorname{cosec}\theta$

10. $\tan\theta$

EXERCISE 5.7

1. 2 **2.** $380\, x^{18}$ **3.** $-x \cos x - 2 \sin x$

4. $-\dfrac{1}{x^2}$ **5.** $x(5 + 6 \log x)$ **6.** $2e^x (5 \cos 5x - 12 \sin 5x)$

7. $9\, e^{6x} (3 \cos 3x - 4 \sin 3x)$ **8.** $-\dfrac{2x}{(1+x^2)^2}$

9. $-\dfrac{(1+\log x)}{(x \log x)^2}$ **10.** $-\dfrac{\sin(\log x) + \cos(\log x)}{x^2}$

12. $-\cot y \operatorname{cosec}^2 y$

Miscellaneous Exercise on Chapter 5

1. $27\,(3x^2 - 9x + 5)^8\,(2x - 3)$ **2.** $3\sin x \cos x\,(\sin x - 2\cos^4 x)$

3. $(5x)^{3\cos 2x}\left[\dfrac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

4. $\dfrac{3}{2}\sqrt{\dfrac{x}{1-x^3}}$ **5.** $-\left[\dfrac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \dfrac{\cos^{-1}\dfrac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$

6. $\dfrac{1}{2}$ **7.** $(\log x)^{\log x}\left[\dfrac{1}{x} + \dfrac{\log(\log x)}{x}\right], x > 1$

8. $(a \sin x - b \cos x) \sin (a \cos x + b \sin x)$

9. $(\sin x - \cos x)^{\sin x - \cos x} (\cos x + \sin x)\,(1 + \log(\sin x - \cos x)), \sin x > \cos x$

10. $x^x (1 + \log x) + a x^{a-1} + a^x \log a$

11. $x^{x^2-3}\left[\dfrac{x^2-3}{x} + 2x \log x\right] + (x-3)^{x^2}\left[\dfrac{x^2}{x-3} + 2x \log(x-3)\right]$

12. $\dfrac{6}{5} \cot \dfrac{t}{2}$ **13.** 0 **17.** $\dfrac{\sec^3 t}{at}, 0 < t < \dfrac{\pi}{2}$

EXERCISE 6.1

1. (a) 6π cm^2/cm (b) 8π cm^2/cm

2. $\frac{8}{3}$ cm^2/s

3. 60π cm^2/s

4. 900 cm^3/s

5. 80π cm^2/s

6. 1.4π cm/s

7. (a) –2 cm/min (b) 2 cm^2/min

8. $\frac{1}{\pi}$ cm/s

9. 400π cm^3/cm

10. $\frac{8}{3}$ cm/s

11. (4, 11) and $\left(-4, \frac{-31}{3}\right)$

12. 2π cm^3/s

13. $\frac{27}{8}\pi(2x+1)^2$

14. $\frac{1}{48\pi}$ cm/s

15. ₹ 20.967

16. ₹ 208

17. B

18. D

EXERCISE 6.2

4. (a) $\left(\frac{3}{4}, \infty\right)$ (b) $\left(-\infty, \frac{3}{4}\right)$

5. (a) $(-\infty, -2)$ and $(3, \infty)$ (b) $(-2, 3)$

6. (a) decreasing for $x < -1$ and increasing for $x > -1$

   (b) decreasing for $x > -\frac{3}{2}$ and increasing for $x < -\frac{3}{2}$

   (c) increasing for $-2 < x < -1$ and decreasing for $x < -2$ and $x > -1$

   (d) increasing for $x < -\frac{9}{2}$ and decreasing for $x > -\frac{9}{2}$

   (e) increasing in (1, 3) and $(3, \infty)$, decreasing in $(-\infty, -1)$ and $(-1, 1)$.

8. $0 < x < 1$ and $x > 2$

12. A, B

13. D

14. $a > -2$

19. D

EXERCISE 6.3

1. (i) Minimum Value = 3 (ii) Minimum Value = – 2
(iii) Maximum Value = 10 (iv) Neither minimum nor maximum value

2. (i) Minimum Value = – 1; No maximum value
(ii) Maximum Value = 3; No minimum value
(iii) Minimum Value = 4; Maximum Value = 6
(iv) Minimum Value = 2; Maximum Value = 4
(v) Neither minimum nor Maximum Value

3. (i) local minimum at $x = 0$, local minimum value = 0
(ii) local minimum at $x = 1$, local minimum value = – 2
local maximum at $x = -1$, local maximum value = 2
(iii) local maximum at $x = \frac{\pi}{4}$, local maximum value = $\sqrt{2}$
(iv) local maximum at $x = \frac{3\pi}{4}$, local maximum value = $\sqrt{2}$
local minimum at $x = \frac{7\pi}{4}$, local minimum value = $-\sqrt{2}$
(v) local maximum at $x = 1$, local maximum value = 19
local minimum at $x = 3$, local minimum value = 15
(vi) local minimum at $x = 2$, local minimum value = 2
(vii) local maximum at $x = 0$, local maximum value = $\frac{1}{2}$
(viii) local maximum at $x = \frac{2}{3}$, local maximum value = $\frac{2\sqrt{3}}{9}$

5. (i) Absolute minimum value = – 8, absolute maximum value = 8

(ii) Absolute minimum value = – 1, absolute maximum value = $\sqrt{2}$

(iii) Absolute minimum value = – 10, absolute maximum value = 8

(iv) Absolute minimum value = 19, absolute maximum value = 3

6. Maximum profit = 113 unit.

7. Minima at $x = 2$, minimum value = – 39, Maxima at $x = 0$, maximum value = 25.

8. At $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

9. Maximum value = $\sqrt{2}$

10. Maximum at $x = 3$, maximum value 89; maximum at $x = -2$, maximum value = 139

11. $a = 120$

12. Maximum at $x = 2\pi$, maximum value = 2π; Minimum at $x = 0$, minimum value = 0

13. 12, 12

14. 45, 15

15. 25, 10

16. 8, 8

17. 3 cm

18. $x = 5$ cm

21. radius = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

22. $\frac{112}{\pi+4}$ cm, $\frac{28\pi}{\pi+4}$ cm

27. A

28. D

29. C

Miscellaneous Exercise on Chapter 6

2. $b\sqrt{3}$ cm²/s

3. (i) $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$ (ii) $\frac{\pi}{2} < x < \frac{3\pi}{2}$

4. (i) $x < -1$ and $x > 1$ (ii) $-1 < x < 1$

5. $\frac{3\sqrt{3}}{4}ab$ **6.** Rs 1000

8. length $= \frac{20}{\pi + 4}$ m, breadth $= \frac{10}{\pi + 4}$ m

10. (i) local maxima at $x = \frac{2}{7}$ (ii) local minima at $x = 2$

(iii) point of inflection at $x = -1$

11. Absolute maximum $= \frac{5}{4}$, Absolute minimum = 1

14. $\frac{4\pi R^3}{3\sqrt{3}}$ **16.** A

SUPPLEMENTARY MATERIAL

CHAPTER 5

Theorem 5 (To be on page 129 under the heading Theorem 5)

(i) Derivative of Exponential Function $f(x) = e^x$.

If $f(x) = e^x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \cdot 1 \quad \left[\text{since } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

Thus, $\dfrac{d}{dx}(e^x) = e^x$.

(ii) Derivative of logarithmic function $f(x) = \log_e x$.

If $f(x) = \log_e x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\log_e (x + \Delta x) - \log_e x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= \frac{1}{x} \quad \left[\text{since } \lim_{h \to 0} \frac{\log_e (1 + h)}{h} = 1\right]$$

Thus, $\dfrac{d}{dx} \log_e x = \dfrac{1}{x}$.

NOTES