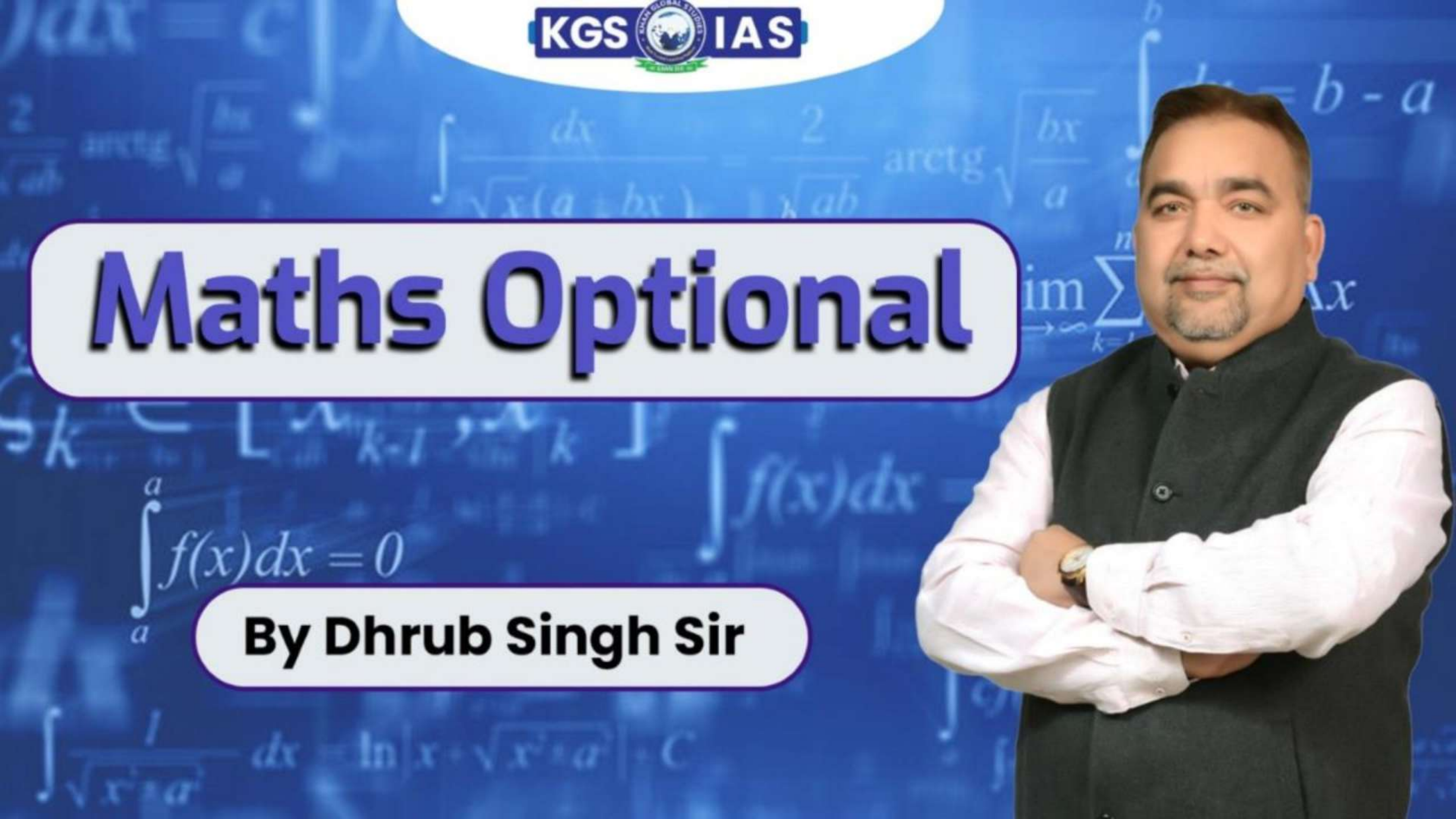
KGS IAS
Maths Optional
By Dhrub Singh Sir

Note ① The row rank of a matrix is the no. of non-zero rows in echelon matrix of the given matrix.

② Row rank = column rank
= Rank of a matrix.

Result: Rank of the sum of two matrices can not exceed the sum of their ranks.

Pre-multiplying with P^{-1} and post-multiplying with Q^{-1}, we have

$$A = P^{-1} \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] Q^{-1}$$

Let us consider another matrix C s.t

$$C = P^{-1} \left[\begin{array}{c|c} O & O \\ \hline O & I_{n-r} \end{array}\right] Q^{-1}$$

Result: If A and B are two n-rowed square matrices then $\text{Rank}(AB) \geq \text{Rank}\,A + \text{Rank}\,B - n$.

Proof: Let $r(A) = r$

$\therefore \exists$ non-singular matrices P and Q s.t.

$$PAQ = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$$

Now

$$r\left[(C+A)\cdot B\right] = r(B)$$

$$r(B) = r[CB + AB]$$

$\Rightarrow r(B) \leq r(CB) + r(AB)$

$\Rightarrow r(B) \leq r(C) + r(AB)$

$\Rightarrow r(AB) \geq r(B) - r(C)$

$\Rightarrow r(AB) \geq r(B) - n + r(A)$

[pre-multiplying by a non-singular matrix doesn't change the rank of a matrix]

$C + A = P^{-1} \cdot I_n \cdot Q^{-1}$

$C + A = P^{-1} \cdot Q^{-1}$

$\because$ P and Q are non-singular.

$\therefore P^{-1}$ and Q^{-1} "

$\therefore P^{-1} \cdot Q^{-1}$ — "

$\therefore C + A$ is non-singular.

$r((C+A)) = n$

Also $r(C) = n - r = n - r(A)$ — ①

Non-Homogenous system of equations

(गैर समरूप)

Consider the system of m non-homogenous equations in n unknowns.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \text{--- (1)}$$

$$AX = B \quad \text{--- (2)}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}, \quad B \neq 0$$

Consistent — System has a soln.

unique

Infinitely many.

Inconsistent — System has no soln.

Augmented matrix: $[A \vdots B]$

Condition for consistency

Result: The equation $AX = B$ is consistent iff $r[A:B] = r[A]$.

Th: If A is a non-singular matrix of order 'n' then the linear system $AX = B$ in 'n' unknowns has a <u>unique</u> sol^n.

proof: $\because A$ is non-singular

$\therefore |A| \neq 0$

and A^{-1} exists.

Also
$$r[A:B] = r[A] = n$$

$\therefore$ System is consistent.

unique solⁿ:

Let X_1 & X_2 be the two solⁿ. of $AX=B$.

$\therefore AX_1=B$ & $AX_2=B$

$\Rightarrow AX_1 = AX_2$

$\Rightarrow A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Rightarrow \underline{(A^{-1}A)}X_1 = (A^{-1}A)X_2$

$\Rightarrow \underline{X_1 = X_2}$

$\because A^{-1}$ exists.

and $AX=B$

$A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow I \cdot X = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1} \cdot B}$

Case (i) $r(A) = r[A : B]$

$= \text{No. of unknowns}$

unique soln.

Case (ii) $r(A) = r[A : B]$

$< \text{No. of var.}$

Infinitely many soln.

Case (iii) $r(A) \neq r(A : B)$

No soln.

Working rule for finding the soln. of $AX = B$

Step 1: Reduce the augmented matrix $[A : B]$ to echelon form by E-row operation.

Step 2: From the echelon form of $[A : B]$, find $r([A : B])$ and $r(A)$.

Step 3: Incase of sol^n,

Write the matrix eq^n. with echelon form. And accordingly solve.

prob: Show that the equations are consistent and hence solve them.

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0.$$

$$AX = B$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$.

writing the system of eqns with echelon form

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{-38}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ -\frac{76}{3} \end{bmatrix}$$

$$-x + 2y + z = 4$$

$$3y + 5z = 16$$

$$\cancel{\frac{-38}{3}} z = \frac{-\cancel{76}^{2}}{\cancel{3}} \Rightarrow z = 2$$

$$3y + 10 = 16 \Rightarrow y = 2$$

$\underline{x = 2}$.

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & \frac{-38}{3} & \vdots & \frac{-76}{3} \end{bmatrix}_{3\times 4}$$

$$R_3 \to R_3 - \frac{7}{3} R_2$$

From this echelon form, we observe that

$$r[A : B] = r[A] = 3$$

$= \text{No. of unknowns}$

System is consistent.

unique soln.

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}_{3\times 4}$$

$R_3 \to R_3 - 3R_2$

we observe that

$r[A:B] = r[A] = 2$

$<$ No. of unkn.

System is consistent

infinitely many

soln.

Prob: Show that the equations

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 7z = 30$$

are consistent and solve them.

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 3 & 6 & \vdots & 24 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

writing system equations with echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$3 - 2 = 1 \rightarrow$ f. vari

$k \in \mathbb{R}$

$x + y + z = 6$

$y + 2z = 8$

let $z = k$, $k \in \mathbb{R}$

$y = 8 - 2k$

$x + 8 - 2k + k = 6$

$x + 8 - k = 6$

$x = k - 2$

Prob: Show that the equations

$$x+y+z=-3$$

$$3x+y-2z=-2$$

$$2x+4y+7z=7$$

are inconsistent.