Conjugate of a matrix

prop: (I) $\overline{(\overline{A})} = A$

(II) $\overline{A+B} = \overline{A} + \overline{B}$

(III) $\overline{KA} = \overline{K} \cdot \overline{A}$

(IV) $\overline{AB} = \overline{A} \cdot \overline{B}$

$A = [a_{ij}]_{m \times n}$

$\overline{A} = [\overline{a_{ij}}]_{m \times n}$

Ex: $A = \begin{bmatrix} 1+i & 1-i \\ 2 & 2+3i \end{bmatrix}$

$\overline{A} = \begin{bmatrix} 1-i & 1+i \\ 2 & 2-3i \end{bmatrix}$

Transposed conjugate ← Tranjugate of a matrix

$$A^{\theta} = (\bar{A})^{T} = \overline{(A^{T})}$$

Ex:
$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \\ 3 & 2-3i \end{bmatrix}_{3\times 2}$$

$$A^{T} = \begin{bmatrix} 1 & 1-i & 3 \\ 1+i & 2 & 2-3i \end{bmatrix}$$

$$A^{\theta} = \overline{(A^{T})} = \begin{bmatrix} 1 & 1+i & 3 \\ 1-i & 2 & 2+3i \end{bmatrix}$$

I $(A^{\theta})^{\theta} = A$.

II $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

III $(kA)^{\theta} = \bar{K} \cdot A^{\theta}$.

IV $(AB)^{\theta} = B^{\theta} \cdot A^{\theta}$.

Hermitian matrix: A square matrix A is called a Hermitian matrix if

$$A^{\theta} = A$$

i.e. $\overline{a_{ij}} = a_{ji} \quad \forall i, j$

Ex: $A = \begin{bmatrix} 2 & 5+i \\ 5-i & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 5-i \\ 5+i & 3 \end{bmatrix}$

$$A^{\theta} = \overline{(A^T)} = \begin{bmatrix} 2 & 5+i \\ 5-i & 3 \end{bmatrix} = A$$

Note: Diagonal element of a Hermitian matrix is real.

$$\overline{a_{ii}} = a_{ii} \quad \forall i$$

It is possible only when a_{ii} is real.

Skew-Hermitian matrix

A square matrix A is called Skew-Hermitian if

$$A^\theta = -A.$$

i.e. $\overline{a_{ij}} = -a_{ji} \ \forall i, j$

Ex: $A = \begin{bmatrix} i & 2+3i \\ -2+3i & 0 \end{bmatrix}$ is Skew-Herm.

$$A^T = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}$$

$$A^\theta = \overline{(A^T)}$$

$$= \begin{bmatrix} -i & -2-3i \\ 2-3i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} i & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$= -A.$$

Note: Diagonal elements of a skew-Hermitian matrix is either purely imaginary or zero.

$$\overline{a_{ii}} = -a_{ii} \quad \forall i$$

$$\Rightarrow \overline{a_{ii}} + a_{ii} = 0 \quad \forall i$$

$(kA)^{\theta} = \bar{k} \cdot A^{\theta}$

Prob: If A is a Hermitian matrix then show that iA is skew Hermitian.

Solⁿ: $\because$ A is Hermitian.

$\therefore$ $A^{\theta} = A$

$$(iA)^{\theta} = \bar{i} \cdot A^{\theta} = -iA$$

$\therefore$ iA is skew-Hermitian.

Prob: If A is a skew-Hermitian matrix then show that iA is Hermitian.

$\longrightarrow$ $\because$ A is skew-Hermitian.

$\therefore A^{\theta} = -A$.

$$(iA)^{\theta} = \overline{i}\, A^{\theta}$$
$$= -i(-A)$$
$$= iA$$

$\therefore$ iA is Hermitian.

Prob: If A and B are Hermitian then show that $AB+BA$ is Hermitian and $AB-BA$ is Skew-Hermitian.

Solⁿ: $\because$ A and B are Hermitian.

$\therefore A^\theta = A, \quad B^\theta = B$

$$(AB+BA)^\theta = (AB)^\theta + (BA)^\theta$$
$$= B^\theta A^\theta + A^\theta B^\theta = BA + AB$$
$$= AB+BA.$$

$$(AB-BA)^\theta$$
$$= (AB)^\theta - (BA)^\theta$$
$$= B^\theta A^\theta - A^\theta \cdot B^\theta$$
$$= BA - AB$$
$$= -(AB-BA)$$

$\therefore AB-BA$ is Skew-Hermitian

<u>prob:</u> If A be any square matrix then prove that $A+A^\theta$, $A\cdot A^\theta$, $A^\theta A$ are all Hermitian matrices and $A-A^\theta$ is skew Hermitian.

$\longrightarrow \quad \underline{(A+A^\theta)}^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A = A + A^\theta$

$(AA^\theta)^\theta = (A^\theta)^\theta \cdot A^\theta = \underline{A \cdot A^\theta}$

$(A-A^\theta)^\theta = A^\theta - (A^\theta)^\theta = A^\theta - A = -(A-A^\theta)$

Prob: Show that the matrix $B^\theta A B$ is Hermitian or Skew-Hermitian according as A is Hermitian or Skew Hermitian.

$\longrightarrow$ $(B^\theta A B)^\theta = B^\theta A^\theta (B^\theta)^\theta$

$= B^\theta A^\theta B$

$= \underline{B^\theta A B}$ if $A^\theta = A$ i.e. A is Herm.

or

$= -B^\theta A B$ if $A^\theta = -A$ i.e. A is Skew-Herm.

Prob: prove that every square matrix is uniquely expressible as the sum of a Hermitian and a skew-Hermitian matrix.

Sol^n: Let A be a square matrix.

$$A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^\theta - \frac{1}{2}A^\theta$$

$$= \underbrace{\frac{1}{2}(A + A^\theta)}_{P} + \underbrace{\frac{1}{2}(A - A^\theta)}_{Q}$$

$P = \frac{1}{2}(A + A^\theta)$

$Q = \frac{1}{2}(A - A^\theta)$

$$P^\theta = \frac{1}{2}(A + A^\theta)^\theta$$

$$= \frac{1}{2}\left[A^\theta + (A^\theta)^\theta\right]$$

$$= \frac{1}{2}\left[A^\theta + A\right]$$

$$= P.$$

$\therefore$ P is Hermitian.

unique

Let $A = R + S$ —— (i)

where R is Hermitian

and S is skew Herm.

(i) $\Rightarrow$

$A^{\theta} = (R+S)^{\theta}$

$\Rightarrow A^{\theta} = R^{\theta} + S^{\theta} = R - S$ —— (ii)

(i) + (ii)

$A + A^{\theta} = 2R$

$\Rightarrow \underline{R = \frac{1}{2}(A + A^{\theta}) = P}$

$Q = \frac{1}{2}(A - A^{\theta})$

$Q^{\theta} = \frac{1}{2}(A - A^{\theta})^{\theta}$

$= \frac{1}{2}[A^{\theta} - (A^{\theta})^{\theta}]$

$= \frac{1}{2}[A^{\theta} - A]$

$= -\frac{1}{2}(A - A^{\theta})$

$= -Q.$

$\therefore$ Q is skew-Hermitian.

$$(\overline{A})^{\theta} = \left(\overline{\overline{A}}\right)^{T}$$
$$= A^{T}$$
$$= \left(A^{\theta}\right)^{T}$$
$$= \left[\left(\overline{A}\right)^{T}\right]^{T}$$
$$= \overline{A}$$

$\therefore \overline{A}$ is Hermitian.

(I) − (II)

$$A - A^{\theta} = 2S$$

$$\Rightarrow S = \frac{1}{2}(A - A^{\theta}) = \phi.$$

prob: prove that $\overline{A}$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

Solⁿ: (1) Suppose A is Hermitian.

$$\therefore A^{\theta} = A \quad \text{—(1)}$$

(ii) A is skew-Hermitian.

$\therefore A^\theta = -A$ —— (ii)

$$
\begin{aligned}
(\overline{A})^\theta &= \overline{(\overline{A})}^T \\
&= A^T \\
&= (-A^\theta)^T \\
&= -\left[(\overline{A})^T\right]^T \\
&= -\overline{A}
\end{aligned}
$$

$\therefore \overline{A}$ is skew-Herm.

$P = \frac{1}{2}(A + A^\dagger)$

$Q = \frac{1}{2i}(A - A^\dagger)$

$P^\dagger = \frac{1}{2}(A + A^\dagger)^\dagger$

$= \frac{1}{2}[A^\dagger + (A^\dagger)^\dagger]$

$= \frac{1}{2}[A^\dagger + A]$

$= P$

$\therefore$ P is Hermitian.

<u>prob:</u> Show that every square matrix A can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices.

$$A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^\dagger - \frac{1}{2}A^\dagger$$

$$= \frac{1}{2}A + \frac{1}{2}A^\dagger + \frac{1}{2}A - \frac{1}{2}A^\dagger$$

$$= \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$$

$$= \underbrace{\frac{1}{2}(A + A^\dagger)}_{P} + i\underbrace{\left[\frac{1}{2i}(A - A^\dagger)\right]}_{Q}$$

uniqueness:

Let $A = R + iS$. —— (I)

where R and S are Hermitian matrices.

$\therefore R^\dagger = R,\ S^\dagger = S.$

(I) $\Rightarrow$ $A^\dagger = [R + iS]^\dagger$

$= R^\dagger + (iS)^\dagger$

$= R^\dagger + \bar{i} S^\dagger$

$A^\dagger = R - iS$ —— (II)

$Q = \frac{1}{2i}(A - A^\dagger)$

$Q^\dagger = \left[\frac{1}{2i}(A - A^\dagger)\right]^\dagger$

$= \frac{1}{\overline{2i}}\left[A^\dagger - (A^\dagger)^\dagger\right]$

$= \frac{1}{-2i}\left[A^\dagger - A\right]$

$= \frac{1}{2i}\left[A - A^\dagger\right]$

$= Q.$

$\therefore Q$ is Herm.

$$(I) + (II)$$

$$A + A^{\theta} = 2R$$

$$R = \frac{1}{2}(A + A^{\theta}) = P$$

$$(I) - (II)$$

$$A - A^{\theta} = 2iS$$

$$\Rightarrow S = \frac{1}{2i}(A - A^{\theta}) = \varphi$$

$$\boxed{A(BC)=(AB)\times C}$$

Prob: If $AB = A$ and $BA = B$ then $B^T A^T = A^T$ and $A^T B^T = B^T$ and hence prove that A^T and B^T are idempotent matrices.

$\because AB = A \Rightarrow (AB)^T = A^T$

$\Rightarrow B^T A^T = A^T$ ——— (i)

$\because BA = B \Rightarrow (BA)^T = B^T$

$\Rightarrow A^T \cdot B^T = B^T$ ——— (ii)

$$(A^T)^2 = A^T \cdot A^T$$

$= A^T \cdot (B^T A^T)$ using (i)

$= (A^T B^T) \cdot A^T$

$= B^T \cdot A^T$ using (ii)

$= A^T$ using (i)

$\therefore A^T$ is idempotent.

$$(B^T)^2 = B^T \cdot B^T$$

$$= B^T \cdot (A^T B^T) \quad \text{using (ii)}$$

$$= (B^T A^T) \cdot B^T$$

$$= A^T \cdot B^T \quad \text{using (i)}$$

$$= B^T \quad \text{using (ii)}$$

$\therefore B^T$ is idempotent.