KGS IAS

Maths Optional

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Th: The rank of a product of two matrices can not exceed the rank of either matrix.

or

If A and B are matrices conformable for multiplication then $r(AB) \le r(A)$ and $r(AB) \le r(B)$.

Proof: Let A and B are matrices of orders $m \times n$ & $n \times p$ resp.

$\therefore$ AB exists

Let $r(A) = r_1$

$r(B) = r_2$

$r(AB) = r$

$\because r(A) = r_1$

$\therefore \exists$ a non-singular matrix P s.t.

$\therefore r(PAB) = r(AB) = r_2$

$\therefore r\left\{\begin{bmatrix} G \\ 0 \end{bmatrix} \cdot B\right\} = r_2$

$\because G$ has r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ 0 \end{bmatrix} \cdot B$ has at most r_1 non-zero rows.

$\therefore r\left\{\begin{bmatrix} G \\ 0 \end{bmatrix} \cdot B\right\} \leq r_1$

$$PA = \begin{bmatrix} G \\ O \end{bmatrix} \quad \text{--- (i)}$$

G is $r_1 \times n$ matrix with rank r_1 and O is $(m - r_1) \times n$ zero matrix.

Post-multiplying both sides of (i) with B, we have

$$PAB = \begin{bmatrix} G \\ 0 \end{bmatrix} B \quad \text{--- (ii)}$$

We know that, pre-multiplying a matrix with a non-singular matrix does not alter the rank of the matrix.

Note:

$$r(AB) \le \min\{r(A), r(B)\}$$

$\therefore r(AB) \le r_1$

i.e. $r(AB) \le r(A)$ — (iii)

Now

$$r(AB) = r\{(AB)^T\}$$
$$= r(B^T \cdot A^T)$$
$$\le r(B^T) = r(B)$$

i.e. $r(AB) \le r(B)$

To find the inverse of a non-singular matrix, using E-row trans.

$A = I \cdot A$

using E-row trans. Identity matrix

same E-row trans. on I

$I = B \cdot A$

$B = A^{-1}$

$A = A \cdot I$

E-col. trans.

same E-col. trans.

$I = A \cdot B$

$B = A^{-1}$

Prob: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

using E-row trans. only.

$\longrightarrow$ **Ans:**

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Prob: Reduce the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix} \text{ to } I_3$$

by a finite sequence of E-row trans. and express A as a product of elementary matrices. Hence find A^{-1}.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \quad \begin{array}{l} E_{21}^{(-4)} \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - R_1 \\ E_{31}^{(-1)} \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \to -R_2 \\ E_2(-1) \end{array}$$

$E_{23}(-5) E_{13}(5) \cdot E_3(1/5)$

$E_{12}(-1) E_2(-1) E_{31}(-1) E_{21}(-4) A = I_3$

$E_{21}(4)$

$A = E_{31}(1) E_2(-1) E_{12}(1) E_3(5) E_{13}(-5) E_{23}(5) I_3$

$A = E_{21}(4) \cdot E_{31}(1) \; E_2(-1) E_{12}(1)$

$E_3(5) E_{13}(-5) E_{23}(5)$

$A^{-1} = E_{23}(-5) E_{13}(5) E_3(\frac{1}{5})$

$E_{12}(-1) \; E_2(-1) E_{31}(-1) E_{21}(-4)$

$$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 5 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$E_{12}(-1)$

$$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \frac{1}{5} R_3$$

$E_3(\frac{1}{5})$

$E_{13}(5)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} &R_1 \to R_1 + 5R_3 \\ &R_2 \to R_2 - 5R_3 \end{aligned}$$

$E_{23}(-5)$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

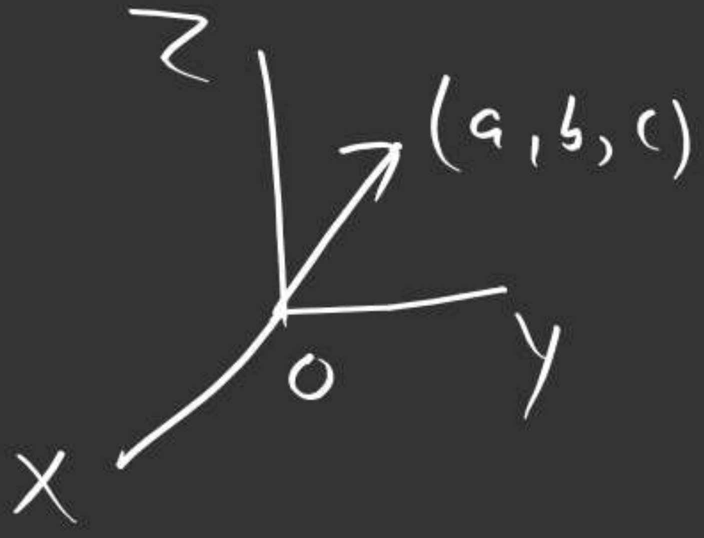
$$= \{(a,b) : a, b \in \mathbb{R}\}$$

$(a \quad b)$ — Row vector

$\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow$ col. vector

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

$$= \{(a,b,c) : a, b, c \in \mathbb{R}\}$$



$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$(a \quad b \quad c)$

$S = \{X, Y\}$.

$L(S) = \{\alpha X + \beta Y ; \alpha, \beta \in \mathbb{R}\}$

Linear span of S

Linear combination of vectors

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $Y = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

L.C. of X & Y

$\alpha X + \beta Y$ $\quad \alpha, \beta \in \mathbb{R}$

$= \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} \alpha - \beta \\ 2\alpha \\ 3\alpha + 2\beta \end{bmatrix}$

Linear independence of vectors: (L.I.)

Ex: $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\alpha X + \beta Y = 0$$

$$\Rightarrow \alpha = \beta = 0$$

X, Y — L.I.

$\alpha, \beta \in \mathbb{R}$

$$\alpha X + \beta Y = 0$$

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \alpha = 0, \ \beta = 0$$

X & Y are L.I.

Basis & Dimension

Ex: $\mathbb{R}^2$

$S = \{X, Y\}$

$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

S is a basis of $\mathbb{R}^2$.

(i) $L(S) = \mathbb{R}^2$

(ii) S is LI.

(i)

$$L(S) = \{aX^T + bY^T : a, b \in \mathbb{R}\}$$

$$= \left\{a\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T : a, b \in \mathbb{R}\right\}$$

$$= \{(a, b) : a, b \in \mathbb{R}\}$$

$$= \underline{\mathbb{R}^2}$$

(ii) S is L.I.

Dimension of $\mathbb{R}^2 = \underline{n(S) = 2}$

Note ① $\underline{\text{Dim } \mathbb{R}^n = n}$.

S – n vectors.
$\underline{L.I.}$

Basis of $\mathbb{R}^n$

S – n vectors.
$\underline{L(S) = \mathbb{R}^n}$

S is a Basis of $\mathbb{R}^n$.

Column space of a matrix

$$A = [c_1 \ c_2 \ - - \ c_n]$$

$$c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j = 1, 2, -- n$$

$$\text{Col. Sp}(A) = \text{Lin. span}\{c_1, c_2, -- c_n\}$$

$$\text{Dim}(\text{Col. Sp}(A)) = \text{Column rank of } A.$$

Row space of a matrix

Let $A = [a_{ij}]_{m \times n}$.

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \quad R_i = [a_{i1} \ a_{i2} - a_{in}]$$

$$i = 1, 2, -- m$$

$$\text{Row space of } A = \text{Lin. Span}\{R_1, R_2, -- R_m\}$$

$$\text{Dim}(\text{Row. SP.}(A)) = \text{Row rank of } A.$$

= No. of non-zero rows of echelon matrix of A.

Note ① Col. Sp. (A) = Row. Sp. (A^T)

② As the non-zero rows of an echelon matrix are L.I.

$\therefore$ $\underbrace{\text{Dim (Row. Sp. (A))}}_{\text{Row rank of A}}$ = Max. No. of L.I. rows of A

= Max no. of L.I. rows of echelon matrix of A.

Row rank of (A^T)
$= 2$

$\therefore$ col. rank of A
$= 2.$

Prob: $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 3 \\ 3 & 1 & -1 & -4 \\ -1 & -2 & -3 & -7 \end{bmatrix}$

Find the column rank of A.

Soln: Col. rank of A = Row rank of A^T.

$$A^T = \begin{bmatrix} 1 & 4 & 3 & -1 \\ 1 & 3 & 1 & -2 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & -4 & -7 \end{bmatrix}$$