Determinants

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}_{n\times n} = [a_{ij}]_{n\times n}$$

Determinant of A is a number.

$\mathrm{Det}(A)$ or Δ or $|A|$ or $|a_{ij}|$

Ex: $\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix}$

$= 1\times 4 - (-2)\times 1$

$= 4 + 2 = 6$

$\rightarrow$ $|a_{11}|_{1\times 1} = a_{11}$. $\quad |-2| = -2 \quad |2| = 2$

$\rightarrow$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\times a_{22} - a_{21}\times a_{12}$.

Minors and cofactors

$a_{ij} \to (-1)^{i+j}$

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} \\ \underset{-}{a_{21}} & \underset{+}{a_{22}} \end{vmatrix}$$

Minor of $a_{11} = M_{11} = a_{22}$

Minor of $a_{12} = M_{12} = a_{21}$

Minor of $a_{21} = M_{21} = a_{12}$

Minor of $a_{22} = M_{22} = a_{11}$

Cofactor of $a_{11} = C_{11} = (-1)^{1+1} \cdot M_{11}$

" " $a_{12} = C_{12} = (-1)^{1+2} \cdot M_{12}$

" " $a_{21} = C_{21} = (-1)^{2+1} \cdot M_{21}$

" " $a_{22} = C_{22} = (-1)^{2+2} M_{22}$

Ex: $|A| = \begin{vmatrix} \overset{+}{3} & \overset{-}{-2} \\ \underset{-}{-4} & \underset{+}{-6} \end{vmatrix}$

$M_{11} = -6$

$M_{12} = -4$

$M_{21} = -2$, $M_{22} = 3$

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{vmatrix}$$

$$C_{11} = M_{11}$$

$$C_{12} = -M_{12}$$

$$C_{13} = M_{13}$$

$$C_{21} = -M_{21}$$

$$C_{22} = M_{22}$$

$$C_{23} = -M_{23}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Expanding through C_2,

$$|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

$$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{33}$$

$$\rightarrow \quad |A| = \begin{array}{c} \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{vmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{array} \end{array}$$

Expanding through R_1

$$|A| = a_{11}\times M_{11} - a_{12}\cdot M_{12} + a_{13}M_{13}$$

$$= a_{11}\times C_{11} + a_{12}C_{12} + a_{13}\cdot C_{13}$$

$$
\begin{array}{cccc}
+ & - & + & - \\
\end{array}
$$

$$
|A| = \begin{vmatrix} 1 & 0 & 5 & 0 \\ 2 & 1 & 2 & 1 \\ -2 & 2 & -3 & -1 \\ 3 & 3 & 4 & 2 \end{vmatrix}
$$

Expanding through R_1

$$
|A| = 1 \begin{vmatrix} 1 & 2 & 1 \\ 2 & -3 & -1 \\ 3 & 4 & 2 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 & 1 \\ -2 & 2 & -1 \\ 3 & 3 & 2 \end{vmatrix}
$$

Ex:

$$
\begin{array}{ccc}
+ & - & + \\
\end{array}
$$

$$
|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix}
$$

Expanding through R_1.

$$
= 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}
$$

$$
= 1 \times (4-2) + 2(2-3) + 3(4-12)
$$

$$
= \cancel{2} - \cancel{2} - 24
$$

$$
= \underline{-24}.
$$

Note: ① $\text{Diag}[d_1\ d_2\ \cdots\ d_n]$

$\Delta = d_1 \cdot d_2 \cdot d_3 \cdots d_n$.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

$|A| = 1 \times (-1) \times 2 \times (-3)$

$= 6$.

② $\text{Det}(I_n) = 1$

③ Det (Triangular Matrix)

= product of diagonal element.

Ex: $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$|A| = 1 \times 4 \times 5$

$= 20$.

Properties of Determinants

Note: $|A| = |A^T|$

① The value of a determinant doesn't change when rows and columns are interchanged.

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Delta_1 = \Delta_2$$

(2) If any two rows (or two columns) of a determinant are interchanged then the value of the determinant will get multiplied with -1.

$\rightarrow$ $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$

$\Delta_1 = -\Delta$

$A_{3\times 3}$ — $|A| = \Delta$

$2A$ — $|2A|$

$= 2^3 \times |A|$

$= 2^3 \times \Delta$

Note: $A_{n\times n}$

$$|KA| = K^n \cdot |A|$$

(3) If all the elements of one row (column) of a determinant are multiplied by some non-zero number K, then the value of the new determinant is K times the value of the original determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ Ka_2 & Kb_2 & Kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K\Delta$$

(4) If two rows (or columns) of a determinant are identical then the value of the determinant will be zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Ex: $\begin{vmatrix} 2 & -2 & 3 \\ 4 & 5 & 6 \\ 6 & -6 & 9 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & -2 & 3 \end{vmatrix}$

$= 3 \times 0$

$= 0.$

5 In a determinant, the sum of the products of the elements of any row (column) with the cofactors of the corresponding elements of any other row (column) is zero.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$a_{11} \times C_{21} + a_{12} C_{22} + a_{13} \cdot C_{23} = 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

Ex:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ {}^{\ominus}3 & 1^{\oplus} & 2^{\ominus} \\ 2 & 1 & 3 \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= -1(6-3) + 2(3-6) - 3(1-4)$$

$$= -3 - 6 + 9 = \underline{0}.$$

(6) If in a determinant each element in any row (column) consists of the sum of two terms, then the determinant can be expressed as the sum of two determinants of the same order.

$$\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$R_1 \to R_1 + mR_2$

7. If to the element of a row (column) of a determinant are added m times the corresponding element of any other row (column), the value of the determinant thus obtained will be same as that of the original det.

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta .$$

Ex:

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 7 & -8 \\ 3 & 10 & -8 \end{vmatrix} \qquad \begin{array}{l} C_2 \to C_2 + 2C_1 \\ C_3 \to C_3 - 3C_1 \end{array}$$

$$= 1 \begin{vmatrix} 7 & -8 \\ 10 & -8 \end{vmatrix} = (-56 + 80)$$

$$= 24.$$

prob: prove that

$$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\Delta = \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = \begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$= 1\begin{vmatrix} b-a & (b-a)(b+a)\\ c-a & (c-a)(c+a)\end{vmatrix}$$

$$= (b-a)(c-a)\cdot\begin{vmatrix}1 & b+a\\ 1 & c+a\end{vmatrix}$$

$$= (b-a)(c-a)\left[c+\cancel{a}-b-\cancel{a}\right]$$

$$= \underline{(b-a)}\,\underline{(c-b)}\,(c-a)$$

$$= \underline{(a-b)(b-c)(c-a)}$$

$\boxed{1+\omega+\omega^2=0}$

Prob: Evaluate

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

where ω is one of the imaginary cube roots of unity.

$$\rightarrow \Delta = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad C_1 \to C_1+C_2+C_3$$

$$= 0.$$