Maths Optional

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σ — E-row trans.

$\sigma(AB) = \underline{\sigma(A)} \cdot B$

σ' — E-col. trans.

$\underline{\sigma'}(AB) = A \cdot \underline{\sigma'(B)}$

Result: Every elementary row transformation over a product AB of two matrices A and B can be obtained by subjecting the pre-factor A to the same elementary row transformation. Similarly every elementary column transformation over a product AB of two matrices A and B can be obtained by subjecting the post factor B to the same elementary column transformation.

$$A = [a_{ij}]_{m\times n} \qquad B = [b_{ij}]_{n\times p}.$$

$$AB_{m\times p}.$$

$$R_1 = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \end{bmatrix}_{1\times n}$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times 1}$$

$$B = \begin{bmatrix} C_1 & C_2 & C_3 & - & - & C_p \end{bmatrix}_{1\times p}$$

$$C_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}_{n\times 1}$$

$$R_i - 1\times n$$

$$C_j - n\times 1$$

$$\underbrace{R_1 \longleftrightarrow R_2}_{6}$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times 1} \qquad B = \begin{bmatrix} C_1 & C_2 & \cdots & C_p \end{bmatrix}_{1\times p}$$

$$\underline{6(AB)} = \begin{bmatrix} R_2C_1 & R_2C_2 & \cdots & R_2C_p \\ R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & R_2C_p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_p \end{bmatrix}_{m\times p}$$

$$= \begin{bmatrix} R_2c_1 & R_2c_2 & \cdots & R_2c_p \\ R_1c_1 & R_1c_2 & \cdots & R_1c_p \\ - & - & & - \\ - & - & & - \end{bmatrix}$$

$$\boxed{\sigma(AB) = \sigma(A)\cdot B}$$

$$\sigma'(AB) = \underline{A\cdot\sigma'(B)}$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \qquad B = \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sigma(A) = \begin{bmatrix} R_2 \\ R_1 \\ \vdots \\ R_m \end{bmatrix}$$

$$\sigma(A)\cdot B = \begin{bmatrix} R_2 \\ R_1 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix}$$

$R_1 \leftrightarrow R_2 \ (\sigma)$

A

$\sigma(A) = \underline{E_{12} \cdot A}$

$C_1 \leftrightarrow C_2 \ (\sigma')$

$\sigma'(A) = \underline{A \cdot E_{12}} \longrightarrow$

__Th:__ Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication (post-multiplication) with corresponding elementary matrix.

$A = [a_{ij}]_{m \times n}$

$A = I_{m \times m} \cdot A_{m \times n}$

$\sigma(A) = \sigma(I \cdot A) = \sigma(I) \cdot A$

$= \underline{E \cdot A}$

σ - E - row trans.

$$A_{m\times n} = A_{m\times n} I_{n\times n}$$

$$\underline{\sigma'(A)} = \sigma'(A\cdot I)$$

$$= A\cdot \sigma'(I)$$

$$= \underline{A\cdot E}$$

σ' — E — col. trans.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\underline{E_2(3)}$$

$$E_2(3) \cdot A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$$

<u>Ex:</u> let $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$

$\sigma : R_2 \to 3R_2$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 6 & 21 & 3 \\ 3 & 8 & 4 \end{bmatrix} =$$

$$\boxed{\sigma(A) = E_2(3) \cdot A}$$

E_{ij} — obtained from I by the El. trans. $R_i \leftrightarrow R_j$

$$E_{ij} \to I$$

$R_i \leftrightarrow R_j$

$$E_{ij} E_{ij} = I$$

Inverses of Elementary matrices

Result: ① $E_{ij}^{-1} = E_{ij}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$I$$

$$E_{12} \cdot E_{12} = I$$

6

Result 2: $\{E_i(k)\}^{-1} = E_i\left(\frac{1}{k}\right)$

6

Result 3: $\{E_{ij}(k)\}^{-1} = E_{ij}(-k)$

$k \neq 0$

$E_2\left(\frac{1}{2}\right) \cdot E_2(2)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $R_2 \to 2R_2$

$R_1 \leftrightarrow R_2$

$r(B) \le r$
$s \le r$

$r \le s$

Theorem: Elementary transformation do not change the rank of a matrix.

$$B = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \longrightarrow A_{m \times n}$$

$r(A) = r$

let $r(B) = s$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$(r+1)$-rowed minor of $B \longrightarrow B_0 \longrightarrow A_0$

$|B_0| = 0$

$|B_0| = |A_0|$

$|B_0| = -|A_0|$

Note: Not every matrix can be reduced to normal form by applying E-row (E-col.) operations alone.

Normal form (First canonical form)

Every non-zero matrix A of rank r can be reduced in one of the following forms:

$$A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \text{ or } \left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right] \text{ or } \left[\begin{array}{c|c} I_r & 0 \end{array}\right]$$

by elementary transformations.

I_r – Identity matrix.

Ex (1) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

can not be reduced to Normal form by E-row trans. only.

(11) $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}$

can not be reduced to normal form by E-col. trans. alone.

Prob: Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

to normal form and hence find its rank.

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \qquad R_2 \to \frac{1}{8} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & 0 \\ 0 & 0 & \frac{9}{4} & -8 \end{bmatrix} \qquad R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{9}{4} & -8 \end{bmatrix} \qquad C_3 \to C_3 - \frac{5}{8} C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix} \qquad C_3 \to \frac{4}{9} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_4 \to C_4 + 8 C_3$$

$$\sim [I_3 \,\vdots\, O]$$

$$\therefore \underline{r(A) = 3}$$

Prob: Reduce the matrix

$$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}_{3\times 4}$$

to normal form and hence find its rank.

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} \quad R_3 \rightarrow R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} \quad \begin{matrix} C_2 \rightarrow C_2 + C_1 \\ C_3 \rightarrow C_3 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \quad R_3 \rightarrow R_3 - 8R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & -4 \end{bmatrix} \quad \begin{matrix} C_3 \rightarrow C_3 - 2C_2 \\ C_4 \rightarrow C_4 - C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad C_3 \to -\frac{1}{12} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_4 \to C_4 + 4C_3$$

$$\left[I_3 \,\vdots\, 0 \right]$$

$$\boxed{r(A) = 3}$$